

1. The “Polyakov-style” action for a massive relativistic point particle, involving the Minkowski space coordinate $X^\mu(\tau)$ and the einbein $e(\tau)$, is given by

$$S = \int d\tau \left(\frac{1}{2} e^{-1} \dot{X}^\mu \dot{X}^\nu \eta_{\mu\nu} - em^2 \right)$$

a. Show that this action has the reparameterization invariance $\tau \rightarrow \tilde{\tau} - \eta(\tau)$, to linear order in η , with

$$\delta e = \frac{d}{d\tau}(\eta(\tau)e) \quad , \quad \delta X^\mu = \frac{dX^\mu}{d\tau}\eta(\tau)$$

b. Consider a path starting at $X^\mu(\tau_1) = X_1^\mu$ and finishing at $X^\mu(\tau_2) = X_2^\mu$. Show that reparameterization invariance allows the choice of gauge

$$e(\tau) = \frac{l}{\tau_2 - \tau_1}$$

where l is the invariant length of the worldline.

c. The Feynman propagator for a massive scalar particle defined by the path integral,

$$G(X_1 - X_2) = \mathcal{N} \int De DX e^{iS[X,e]}$$

with \mathcal{N} the usual normalization constant. By transforming to the gauge $e = l/(\tau_2 - \tau_1)$, together with an appropriate redefinition of the parameter τ , show that

$$G(X_1 - X_2) = \mathcal{N}' \int_0^\infty dl DX \exp \left(\frac{i}{2} \int_0^l d\tau (\dot{X}^2 - m^2) \right)$$

Why does the integral over the length of path l remain? (You may find the discussion in Polchinski chapter 5.1 useful to answer this).

d. Compute the functional integral over X to deduce that the propagator in momentum space is given by

$$\tilde{G}(p) = \frac{1}{p^2 + m^2}$$

2. The scattering amplitude for m closed string tachyons is given by,

$$\mathcal{A}^{(m)}(p_1, \dots, p_m) = \frac{g_s^{m-2}}{\text{Vol}(SL(2; \mathbf{C}))} \int \prod_{i=1}^m d^2 z_i \langle \hat{V}(z_1, p_1) \dots \hat{V}(z_m, p_m) \rangle$$

where $\hat{V}(z, p) = e^{ip \cdot X(z, \bar{z})}$ and the correlation function is computed using the gauge fixed free Polyakov action

$$S_{\text{Poly}} = \frac{1}{2\pi\alpha'} \int d^2 z \partial X \cdot \bar{\partial} X$$

a. By expressing the correlation function as a Gaussian integral, show that the amplitude is given by

$$\mathcal{A}^{(m)} \sim \frac{g_s^{m-2}}{\text{Vol}(SL(2; \mathbf{C}))} \delta^{26}(\sum_i p_i) \int \prod_{i=1}^m d^2 z_i \prod_{j<l} |z_j - z_l|^{\alpha' p_j \cdot p_l}$$

b. Show that this integral is invariant under the $SL(2; \mathbf{C})$ transformation

$$z_i \rightarrow \frac{az_i + b}{cz_i + d}$$

only when the momenta are on-shell, i.e. $p_i^2 = 4/\alpha'$.

c. Explain why this means that the 4-point amplitude can be reduced to the integral

$$\mathcal{A}^{(4)} \sim g_s^2 \delta^{26}(\sum_i p_i) \int d^2 z |z|^{\alpha' p_2 \cdot p_3} |1 - z|^{\alpha' p_3 \cdot p_4}$$

d. Evaluate this integral in terms of gamma functions. Show that, when written in Mandelstam variables, it is given by the Virasoro-Shapiro amplitude

$$\mathcal{A}^{(4)} \sim g_s^2 \frac{\Gamma(-1 - \alpha' s/4) \Gamma(-1 - \alpha' t/4) \Gamma(-1 - \alpha' u/4)}{\Gamma(2 + \alpha' s/4) \Gamma(2 + \alpha' t/4) \Gamma(2 + \alpha' u/4)}$$

3. Explain why the limit $s \rightarrow \infty$, with t fixed corresponds to small angle scattering at high energy. Show that in this limit the Virasoro-Shapiro amplitude exhibits so-called Regge behaviour,

$$\mathcal{A}^{(4)} \rightarrow g_s^2 \delta^{26}(\sum_i p_i) \frac{\Gamma(-1 - \alpha' t/4)}{\Gamma(2 + \alpha' t/4)} s^{2 + \alpha' t/2}$$

4a. Write down the vertex operator for a massless closed string state with polarization $\zeta_{\mu\nu}$ and momentum p^μ . What are the restrictions on p^μ and $\zeta_{\mu\nu}$?

b. Consider the scattering of a massless closed string mode with momentum p_1 and two tachyons with momentum p_2 and p_3 . Show that $p_1 \cdot p_2 = p_1 \cdot p_3 = 0$ and $p_2 \cdot p_3 = -4/\alpha'$.

c. Show that the 3-point scattering amplitude for these particles is given by

$$\mathcal{A}^{(3)} \sim \frac{g_s}{\text{Vol}(SL(2; \mathbf{C}))} \delta^{26}(\sum_i p_i) \int \prod_{i=1}^3 d^2 z_i \frac{1}{|z_{23}|^4} \zeta_{\mu\nu} \left(\frac{p_2^\mu}{z_{12}} + \frac{p_3^\mu}{z_{13}} \right) \left(\frac{p_2^\nu}{\bar{z}_{12}} + \frac{p_3^\nu}{\bar{z}_{13}} \right)$$

where $z_{ij} = z_i - z_j$.

d. Explain why the $SL(2; \mathbf{C})$ gauge symmetry allows us to simplify this to

$$\mathcal{A}^{(3)} \sim g_s \delta^{(26)}(\sum_i p_i) \zeta_{\mu\nu} (p_2^\mu - p_3^\mu) (p_2^\nu - p_3^\nu)$$

5a. After using $SL(2; \mathbf{C})$ to fix the positions of 3 vertex operators, the tree-level m -point amplitude for tachyon scattering reduces to an integral over the positions of the remaining $m - 3$ vertex operator insertions,

$$\mathcal{A}^{(4)} \sim g_s^{m-2} \delta^{26}(\sum_i p_i) \int \prod_{i=4}^m d^2 z_i \prod_{j<l} |z_{jl}|^{\alpha' p_j \cdot p_l}$$

where $z_{jl} = z_j - z_l$. The variables describing the exchange of momentum are $s_{ij} = -(p_i + p_j)^2$. The hard scattering limit is defined by $s_{ij} \rightarrow \infty$. Explain why the integral can be evaluated using a saddle-point approximation in this limit.

b. For the 4-point amplitude, use the saddle point approximation to show that

$$\mathcal{A}^{(4)} \sim g_s^2 \delta^{26}(\sum_i p_i) \exp \left(-\frac{\alpha'}{2} (s \ln s + t \ln t + u \ln u) \right)$$