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Abstract of Dissertation Presented to the Graduate School of the University of Florida in Partial Fulfillment of the Requirements for the Degree of Doctor of Philosophy

SISTER TRAJECTORIES IN STRING THEORY

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It is shown that by using higher-order corrections that include sister trajectories, it may be possible to restore the Cerulus-Martin bound in string theory, which would resolve an objection to locality. In the single-Regge limit, we review the existence of the first sister trajectory in the six-point function and then exhibit the second sister in the eight-point function. New work includes demonstrating that the first sister enters the four-point function at two loops, and that it can be seen across the intermediate open string propagator which appears as a higher order correction to closed string four-point scattering.

We then introduce a procedure for determining the state representation of the sister trajectories, $a_m(t)$, for space-like momentum transfer squared $t$. These sister states are obtained by analytically continuing from the physical states, and involve reciprocal oscillators signalling the sisters unphysical nature. We consider both tree level and higher order scattering.
A major hope of string theory has been that it would describe physics at the Planck scale. This entails understanding its short distance behavior and considering the related issue of locality. The latter is important because if the theory is nonlocal at a fundamental level then acausality may result, which is probably unacceptable. The high energy behavior at tree level seems to suggest that it is not local, although Gross and Mende\textsuperscript{1} claim that it still obeys causality because the interaction of strings is local. The tree level objection to locality in string theory is that in the limit of high center of mass energy, $s \to \infty$, for fixed scattering angle $\sin^2(\phi/2) \approx -\frac{1}{s}$, it does not behave as do theories for nonextended objects. Cerulus and Martin\textsuperscript{2}(CM) found that for general theories, under certain assumptions which include locality, the scattering amplitude obeys the rigorous lower bound $|A(s, t)| \geq e^{-\sqrt{s} \ln se(\theta)}$. String theory, on the other hand, has the tree level fixed-angle behavior $|A(s, t)| \to e^{-sf(\theta)}$, which was pointed out even in Veneziano’s original paper.\textsuperscript{3} Gross and Mende\textsuperscript{1} and later Mende and Ooguri,\textsuperscript{4} attempted to determine if stringy perturbative corrections could restore the CM bound, but were unable to control the higher order corrections to reach a decisive conclusion.

The CM bound was derived using the assumptions of unitarity, existence of a finite mass gap, and polynomial boundedness. Off-shell, covariant string theory must introduce Fadeev-Popov ghosts to be unitary. However, unitarity can be established on-shell in the critical dimension, for instance, by breaking
manifest Lorentz invariance and fixing to the light-cone gauge. String theory also does not have a finite mass gap since it has massless particles. But this probably does not lead to the violation of the CM bound. The final assumption of uniform polynomial boundedness states that, for fixed $t$, the amplitude $|A(s, t)|$ is bounded by $s^N$ where $N$ does not depend on $s$ or $t$. String theory does not obey this condition either for $t > N$ since its fixed $t$ behavior goes as $s^t$. Nor does this behavior obey even the weaker condition where $N \sim O(\sqrt{t})$ which Martin showed also gives the CM bound.\(^5\) Furthermore, in quantum field theory, polynomial boundedness is a consequence of locality. As a result, it is generally thought this power behavior of $s$ leads to the CM bound violation.\(^1,6\) Although restoring the CM bound is necessary for a local string theory, it is not sufficient. Nevertheless, resolving this issue may be important for future development of string theory.

In this thesis we will show how stringy corrections can slow the exponential fall-off of the scattering amplitude for $t \to -\infty$. However, unlike Gross and Mende who examined the high energy behavior that dominates over the entire moduli space, we will focus on particular processes that dominate just a small region. Our analysis will expose an overall subdominant behavior that is consistent with the lower limit of the CM bound. This suggests that the more dominant behavior considered by Gross and Mende may actually exceed the lower bound. However, we have yet to sum the perturbative series to verify if in fact the bound is obeyed. By considering subdominant behavior we hope that in performing the perturbative sum it may be possible to avoid the uncontrollable corrections which obstructed Gross and Mende.

The plan is as follows. In Chap. 2 we review the emergence of the linear Regge trajectories, called sisters, which have more gradual slopes than the
standard $a(t)$ Regge trajectories at large negative transfer momentum squared $t$. Rather than working directly in the fixed angle limit, we find it more convenient to take $s \to \infty$ with $t$ held fixed. We show that the complete set of sister trajectories is consistent with the CM bound. We then present a slightly new approach for obtaining the sister contributions and discuss six-point and eight-point tree level scattering. The generalization to $N$-point scattering should then be apparent. Since the sister trajectories occur in the bosonic sector, our results equally apply to the Superstring and Heterotic string.

In Chap. 3 we will show that sisters first appear in the open string four-point scattering amplitude at the double-loop level. We focus on this process because four-point scattering is the simplest case which must be shown to obey the CM bound. An interesting result suggested from our analysis is that the sisters also occur in the non-planar case defined when twists are placed on both loops while they are separated by the intermediate sister propagator. Thus, in Chap. 4 we are led to consider the case of four interacting closed strings with an intermediate open string propagator. We will find that the open string propagator supports the sister trajectory, which is degenerate with the dilaton trajectory that may appear on the connecting closed string propagators.

In Chap. 5 the focus is shifted to the the state interpretation of the sisters. We will determine the oscillator representation of the sisters by isolating the appropriate propagator, and then saturating it with string oscillators. We then analytically continue to the unphysical sister state. This procedure also affords us the opportunity to confirm the amplitudes derived in the high energy analysis. In Chap. 6 we give some concluding remarks and discuss a possible physical interpretation of the sisters.
In the appendix we discuss the weight diagram construction of Lax operators, which have no connection with sister trajectories. Lax operators have gained recent popularity in their application to the theory of matrix models, which have been shown to be related to low dimensional string theories.\textsuperscript{7,8,9}

Each Lax operator can be associated with a particular representation of an affine Lie algebra, and generates a corresponding KdV equation. These KdV equations, in turn, generate integrable systems which can reproduce matrix model results.

Our notation is as follows. The standard Regge trajectory is given by $a(t) = \alpha' t + \alpha_0$, where we choose the open string slope $\alpha' = 1$ and intercept $\alpha_0 = 1$. This leads to a tachyon mass of $m^2 = -1$. In the same units, for the closed string we have $a(t) = \frac{1}{2} t + 2$. The trajectory $a(t_i)$ is associated with the momentum transfer squared $t_i$ across the propagator $z_i$. Finally, the trajectory $a(s_{ij})$ is defined with respect to the energy $s_{ij} \equiv -(p_i + p_{i+1} + \cdots + p_j)^2$. 
CHAPTER 2
SISTER TRAJECTORIES

A central feature of the dual resonance theory\(^3\) was that in the high energy limit \(s \to \infty\) the scattering amplitude scales as \(A(s, t) \propto s^{\alpha(t)}\), for fixed \(t\). Until the mid-70’s, it was thought that the Regge trajectories \(\alpha(t)\) were linear and parallel, \(i.e., \alpha(t) = \alpha't + \alpha_0\), differing only in their intercept \(\alpha_0\). Then, in 1976, Hoyer, Törnqvist and Webber\(^{10}\) discovered that the theory also predicted a new "sister" trajectory with a slope half that of the leading Regge trajectory. They were led to this result by a careful examination of the six particle scattering tree amplitude of Fig. 1.

Hoyer \textit{et al.} argued as follows. In the limit \(s \to \infty\), the six-point amplitude factorizes as follows:

\[
A_6 \simeq D(\alpha_a)V(\alpha_a, \alpha_b)D(\alpha_b)V(\alpha_b, \alpha_c)D(\alpha_c). \tag{2.1}
\]

Here, the propagators \(D(\alpha)\) have zeros for \(\alpha = -1, -2, \ldots\). On the other hand, the vertices \(V(\alpha_a, \alpha_b)\) have unphysical poles for \(\alpha_a, \alpha_b = -1, -2, \ldots\), which have the undesirable properties of negative spin\(\text{(nonsense)}\) and wrong-signature. Now, for \(\alpha_b\) there is only one zero coming from the central propagator, while there are two poles coming from the adjacent vertices. This leaves an unphysical pole, which does not appear in the exact expression for the scattering amplitude. For the theory to be consistent there must be some mechanism to cancel this unwanted pole.
Figure 1. The first sister $\beta(t)$ requires twists on both $z_1$ and $z_3$.

At that time, most people worked with four-dimensional models. Although string theories are simplest in critical space-time dimensions of 10 and 26, we can still adopt this restriction in the high energy limit by permitting momentum to grow large in only four dimensions. In this case, for six particle scattering, there are eight kinematic degrees of freedom yet nine free parameters. Thus, one must apply a four-dimensionality constraint.\textsuperscript{11} For the higher dimension string theories, one would have to consider a higher N-point function with corresponding dimensionality condition. The following discussion holds in any case.

To examine the high energy behavior of the amplitudes in a way that makes sense requires that one first analytically continue the energy into the complex plane, \textit{e.g.}, $s \to i\infty$.\textsuperscript{12} What Hoyer \textit{et al.} observed was that previous analyses had imposed the dimensionality condition only after the energy had been analytically continued back to the physical plane. The effect of not maintaining the constraint throughout the calculation is that some critical point remains hidden. Fixing this oversight and imposing the constraint before the back continuation, allows a factorization to occur in the amplitude, which then
exposes the critical point. Integration about this point then leads to the behavior \( |A(s, t)| \rightarrow s^{\beta(t)} \) where \( \beta(t) = \frac{1}{2} \alpha(t) - \frac{1}{2} \) is the first sister trajectory, for \( \beta(t) = -1, 0, 1, \ldots \) or equivalently \( \alpha(t) = -1, 1, 3, \ldots \). Furthermore, at \( \alpha(t) = -1 \), it was explicitly shown that the pole due to the sister trajectory \( \beta(t) \) precisely cancels the remaining unphysical pole coming from the \( \alpha(t) \) trajectory. In addition, the sister trajectory has associated daughters which cancel the poles at \( \alpha(t) = -2, -3, \ldots \).

Ref. 10 also noted that the sister trajectory had been elusive in the past because at each vertex it can not couple to more than one on-shell state. This decoupling can easily be understood by considering the factorization (2.1). In the high energy limit, the end vertices that couple an intermediate propagator to two on-shell states are represented by factors of unity, which obviously do not have poles.

Shortly after the discovery of the first sister in the six-point amplitude, working in the helicity-pole limit, Hoyer\textsuperscript{13} showed that the eight-point tree amplitude predicts a second sister trajectory \( \gamma(t) = \frac{1}{3} \alpha(t) - 1 \). The purpose of this second sister is to cancel unphysical poles occurring on the \( \beta(t) \) sister trajectory for \( \beta(t) = -2, -3, \ldots \). He then proposed the generalization,

\[
\alpha_m(t) = \frac{1}{m} \alpha(t) - \frac{1}{2} (m - 1),
\]

where the \( m^{\text{th}} \) sister first appears at tree level for \( 2m + 4 \) interacting particles. Hoyer et al. then showed that the first sister \( \beta(t) \) could be obtained using the more general single-Regge limit,\textsuperscript{14} as opposed to the helicity-pole limit used in Ref. 10. Other work followed which examined the sister trajectories under the, even less general, multi-Regge limit.\textsuperscript{15,16}

Sisters were subsequently found in the Neveu-Schwarz sector of the NSR superstring,\textsuperscript{17} and related phenomenological implications were discussed.\textsuperscript{18}
Figure 2. Plot showing the leading $\alpha(t)$ Regge trajectory and the first two sisters, $\beta(t)$ and $\gamma(t)$.

Quirós showed that the sister $\beta(t)$ appears in the single-loop six-point diagram, and that it renormalizes the corresponding tree level sister.\textsuperscript{19} Further, several papers also considered the closed bosonic string which found that, as in the open string, sisters appear at tree level when there are at least six interacting particles.\textsuperscript{20,21,22}

The first two sister trajectories are shown in Fig. 2 along with the leading Regge trajectory. Due to successively more gradual slopes, the net behavior of the sisters is clearly not linear as $t \to -\infty$. We can find the asymptotic behavior by considering the intersection of two neighboring curves $\alpha_m(t)$ and $\alpha_{m+1}(t)$ and then letting $m \to \infty$. Equating these using (2.2), we easily find that to lowest order $\alpha(t) \simeq -\frac{1}{2}m^2$. Comparing this with (2.2) gives

$$\alpha_m(t) \simeq -m = -\sqrt{-2a(t)} \approx -\sqrt{-t}.$$ \hfill (2.3)
For fixed angle scattering we conclude $s^{\alpha_m(t)} \rightarrow s^{-\sqrt{s}}$, which is the CM bound. This short calculation also demonstrates why locality may be responsible for the CM bound violation. In string theory, the fundamental length scale is defined by $l = \sqrt{\alpha'}$. Further, the limit $m \rightarrow \infty$ is completely equivalent to $\alpha' \rightarrow 0$. Thus, the length scale $l$ associated with each sister approaches zero as the order of the sister increase. This means that each successive sister appears more local than the previous one, and in the asymptotic limit we reach point-like behavior.

The possibility for restoring the CM bound in four-point scattering exists if we can show that the entire set of sisters, $\alpha_m(t)$, is present. Since, at each vertex, sisters do not couple to more than one on-shell state, we must consider higher order corrections. We make an initial step in this direction by showing that the first sister $\beta(t)$ couples at the double-loop level. Because $t$ is to be held fixed, and in order to work under the most general conditions, we apply the single-Regge limit $s \rightarrow \infty$. However, since the original approach found in Appendix B of Ref. 14 requires a priori knowledge of any twists, we modify the calculation to remove the need for their explicit presence in the initial expression of the amplitude. This has the advantage of allowing us to consider many cases simultaneously, which significantly reduces the amount of work evaluating higher order functions. Note that in the multi-Regge limit, one can only determine the need for twists by first inserting them, and then computing the final result to see if sisters appear.

**Tree Level Six-Particle Scattering**

The sister trajectory $\beta(t)$ is seen in the high energy limit of the six-point function only if twists are placed on both of the adjoining propagators as shown
in Fig 1. We begin, however, with the corresponding untwisted amplitude. This is easily calculated in the Fubini-Veneziano formalism from

\[ A_6 = \langle 0, p_1 | V(p_2) \Delta V(p_3) \Delta V(p_4) \Delta V(p_5) | 0, p_6 \rangle. \]  

(2.4)

In general, passing the vertex operators through each other produces factors of the form

\[ \exp\left[ -2 \sum_{n=1}^{\infty} \frac{z^n}{n} \right] \equiv (1 - z)^{2 p_i \cdot p_j}, \]

(2.5)

where \( z \) is the product of coordinates \( z_i \) which are associated with the propagators connecting the vertices. Final expressions for the amplitude are usually written in terms of the right-hand factors. The left-hand form is more convenient, however, for locating critical points in the high energy limit \( s \to i\infty \). Consequently, we use the left-hand side of (2.5) if one of the connecting propagators sees the energy \( s \), and the right-hand side for non-overlapping quantities. In the particular case of Fig. 1, the complete exponential factor is then easily found to be

\[ \exp\left[ 2 \sum_{n=1}^{\infty} \frac{z^n}{n} (-p_2 z_1^n - p_3) \cdot (p_4 + p_5 z_3^n) \right]. \]

(2.6)

Substituting in the momentum scalar products, the full amplitude becomes

\[ A_6 \simeq \int_0^1 dz_1 dz_2 dz_3 z_1^{-1 - \alpha(t_1)} z_2^{-1 - \alpha(t_2)} z_3^{-1 - \alpha(t_3)} \]

\[ \times (1 - z_1)^{-1 - \alpha(s_{23})} \]

\[ \times (1 - z_3)^{-1 - \alpha(s_{45})} \]

\[ \times \exp\left[ \sum_{n=1}^{\infty} \frac{z_2^n}{n} \left( z_1^n (s_{24} - s_{34}) + z_1^n z_3^n \tilde{s} + s_{34} + z_3^n (s_{35} - s_{34}) \right) \right], \]

(2.7)

where we have defined

\[ \tilde{s} = s_{34} + s_{61} - s_{24} - s_{35}. \]

(2.8)

In writing (2.7), we have also dropped terms in the exponential which can be safely neglected in the high energy limit.
We are now in a position to impose the four-dimensionality constraint, which, in the high energy limit, reduces to

\[
\frac{s_{35} s_{24}}{s_{34} s_{61}} = 1. \tag{2.9}
\]

Applying this constraint to (2.7) allows the argument of the exponential to be factorized, giving

\[
A_6 \simeq \int_0^1 dz_1 dz_2 dz_3 z_1^{-1 - \alpha(t_1)} z_2^{-1 - \alpha(t_2)} z_3^{-1 - \alpha(t_3)} (1 - z_1)^{-1 - \alpha(s_{23})} \times (1 - z_3)^{-1 - \alpha(s_{45})} \exp \left[ \tilde{s} \sum_{n=1}^\infty \frac{z_2^n}{n} (z_1^n - x_1)(z_3^n - x_3) \right], \tag{2.10}
\]

where

\[
x_1 = \left(1 - \frac{s_{61}}{s_{35}}\right)^{-1}, \quad x_3 = \left(1 - \frac{s_{61}}{s_{24}}\right)^{-1}. \tag{2.11}
\]

To discuss the high energy limit we must let \( \tilde{s} \to \infty e^{i \delta} \) where the real part of \( \tilde{s} \) is held fixed, and \( \delta \) is such that the real part of \( s \) is in the strip of convergence.\(^{14}\) The result is a Fourier integral whose asymptotic behavior is dominated by its critical points.\(^{12}\) For \( (x_1, x_3) \) to be a useful critical point it must fall within the integration region, \( 0 < z_1, z_3 < 1 \). Critical points taken at the boundaries do not produce sisters. Since the boundary of the integration region is not included, the factors in (2.7), other than the exponential, can be ignored during integration. To recover the proper limit \( \tilde{s} \to \infty e^{i \delta} \), we obtain a double critical point by choosing the phases

\[
s_{34}, s_{61} \to \infty e^{i \delta}, \tag{2.12}
\]

\[
s_{24}, s_{35} \to -\infty e^{i \delta}.
\]

This is completely equivalent to twisting the propagators corresponding to \( z_1 \) and \( z_3 \) since energies that overlap an odd number of twisted propagators change sign. In other words, the role of the twists here is to place the critical point inside the integration region.
To obtain the leading sister trajectory by evaluating (2.7) about the critical points, we keep only the lowest order terms in the exponential and integrate about

\[ |z_1 - x_1| \leq \epsilon, \quad |z_3 - x_3| \leq \epsilon. \]  

(2.13)

for \( \epsilon \) small. Choosing higher powers of \( z_1 \) and \( z_3 \) would lead to daughter trajectories. After shifting \( z_1 \) and \( z_3 \), we obtain

\[ A_6 \sim x_1^{-1-a(t_1)} x_3^{-1-a(t_3)} (1 - x_1)^{-1-a(s_{23})} (1 - x_3)^{-1-a(s_{45})} I_6. \]  

(2.14)

where

\[ I_6 = \int_0^1 dz_2 z_2^{-1-a(t_2)} e^\hat{s} z_2^2 c \int_{-\epsilon}^{\epsilon} dz_1 d z_3 \exp[\hat{s} z_2 z_1 z_3]. \]  

(2.15)

and

\[ c = \frac{1}{2} x_1 x_3 (x_1 - 1)(x_3 - 1). \]  

(2.16)

Setting \( y = -ie z_2 z_3 \hat{s} \), gives

\[ I_6 = i(e \hat{s})^{-1} \int_0^1 dz_2 z_2^{-2-\alpha(t_2)} e^\hat{s} z_2^2 c \int_{-\epsilon}^{\epsilon} dz_1 \int_{-y_0}^{y_0} dy \exp(i e^{-1} y z_1). \]  

(2.17)

where \( y_0 = -i e^2 z_2 \hat{s} \). Integrating over \( z_1 \) we easily find

\[ I_6 = \hat{s}^{-1} \int_0^1 dz_2 z_2^{-2-\alpha(t_2)} e^\hat{s} z_2^2 c \int_{-y_0}^{y_0} dy \frac{\exp(i y) - \exp(-i y)}{y}. \]  

(2.18)

where the \( y \) integral is symmetric. If \( z_3 \) were not critical, taking the limit \( \epsilon \to 0 \) now would give \( I_6 = 0 \) (use \( dy \sim \epsilon \)). This demonstrates the need for a double critical point.

Now, define \( z = -\hat{s} z_2^2 c \), which gives

\[ I_6 = -(-\hat{s} c)^{\frac{1}{2}} a(t_2) + \frac{1}{2} \hat{s}^{-1} \int_0^\infty dz z^{-\frac{3}{2} - \alpha(t_2)} e^z \int_{-y_0}^{y_0} dy \frac{\exp(i y) - \exp(-i y)}{y}. \]  

(2.19)

For us to consistently write \( z = y^2 e^{-4 \hat{s}^{-1}} c \) we must have \( 1 > \epsilon > \hat{s}^{-\frac{1}{4}} \) to reach the lower limit \( z \to 0 \) for fixed \( y \). Consequently, in the high energy limit
\[ s \rightarrow i\infty, \ y_0 \rightarrow \infty \] and so the integration over \( y \) gives \( i\pi \). Next, the \( z \) integral gives \( \Gamma(-\frac{1}{2} - \frac{1}{2} \alpha(t_2)) \) which is valid only for \( \alpha(t_2) < -1 \). Thus, the complete amplitude is

\[
A_6 \sim -i\pi(-s)\frac{1}{2} + 1/2 \alpha(t_2) s - 1 \Gamma(-\frac{1}{2} - \frac{1}{2} \alpha(t_2)) x_1^{1-\alpha(t_1)} x_3^{1-\alpha(t_3)} \\
\times (1 - x_1)^{-1-\alpha(s_{23})} (1 - x_3)^{-1-\alpha(s_{35})}.
\]

(2.20)

We now analytically continue the energy back by making the replacement \(-s \rightarrow e^{-i\pi s}\). Defining \( \beta(t_2) = \frac{1}{2}\alpha(t_2) - \frac{1}{2} \), which corresponds to the first sister trajectory, and simplifying, we finally arrive at

\[
A_6 \sim i\pi e^{-i\pi \beta(t_2)} 2^{-\beta(t_2)} - 1 s \beta(t_2) \Gamma(-\beta(t_2) - 1) x_1^{\beta(t_2) - \alpha(t_1)} x_3^{\beta(t_2) - \alpha(t_3)} \\
\times (1 - x_1)^{\beta(t_2) - \alpha(s_{23})} (1 - x_3)^{\beta(t_2) - \alpha(s_{35})}.
\]

(2.21)

Since each of the energies comprising \( s \) overlaps with \( s_{34} \), the Regge behavior \( s \beta(t_2) \) shows that the central propagator in Fig. 1 sees the sister. Using the four-dimensionality constraint (2.9) we can write

\[
\tilde{s} = s_{34} + s_{61} - s_{24} - s_{35} = s_{61}^{-1}(s_{35} - s_{61})(s_{24} - s_{61}),
\]

(2.22)

and easily recover Eq. B.19 of Ref. 14.

Examining the \( \Gamma \) function in (2.21), we see that the poles of the sister trajectory are for \( \beta(t_2) = -1, 0, 1, \ldots \). Our approach makes it particularly easy to determine the signature \( \tau \) of these poles. Twisting the sister propagator \( t_2 \) changes the sign of all overlapping energies. Although both the numerator and denominator of \( x_1 \) and \( x_3 \) change sign in Eq. (2.11), the signs of the energy ratios remain unchanged. Thus, the twisted and untwisted diagrams can be added together giving an overall factor \( \tau + 1 \). Therefore, the poles of \( \beta(t_2) \) have pure positive signature. Since these poles correspond to odd values of spin, \( i.e., \alpha(t_2) = -1, 1, 3, \ldots \), they have unphysical wrong-signature.

For the existence of the sister it was necessary that the argument of the exponential factorize, producing a 2-tuple critical point. Integrating over both
Figure 3. At tree level, the second sister $\gamma(t)$ first appears in eight-point scattering. Concurrently, $z_2$ and $z_4$ see $\beta(t)$.

coordinates, in effect, removed the linear power of the propagator variable $z_2$ from the exponential. In general, integrals of the form

$$I = \int_0^1 z^{-\alpha-1-n} \exp(-cz^m)$$

(2.23)

in the limit $c \to \infty$ integrate to

$$I \simeq \frac{1}{m} \Gamma\left(-\frac{\alpha}{m} - \frac{n}{m}\right) c^{\frac{\alpha}{m} + \frac{n}{m}} \quad \text{for} \quad \alpha < -n.$$  

(2.24)

Thus, sisters do not appear in 4- or 5-pt scattering since both retain the linear power of $z$. Furthermore, to produce the second sister $\gamma(t)$, both the linear and quadratic powers of $z$ must be integrated away, leaving the cubic power. This occurs when the critical point is a 4-tuple, which first arises in the eight-point scattering amplitude.

Tree Level Eight-Particle Scattering

In this section we will expose the second sister, $\gamma(t)$, in the open string tree diagram of Fig. 3, where the sister appears across the propagator with $z_3$. In the corresponding amplitude, we isolate the relevant terms by including in the exponential only quantities which overlap the central propagator. We gather the other terms into a function $f(z_1, z_2, z_4, z_5)$, whose exact form can
be ignored since, as shown in the last section, the sisters depend only on the exponential factor. The advantage of using the single-Regge limit over past approaches becomes more apparent in this example.

From Fig. 3(without twists), we immediately write down

$$A_8 = \int_0^1 \prod_{i=1}^5 dz_i f(z_1, z_2, z_4, z_5) z_3^{-1-\alpha(t_3)}$$

$$\times \exp \left[ 2 \sum_{n=1}^{\infty} \frac{z_n^3}{n} \left( -p_2 z_1^n z_2^n - p_3 z_2^n - p_4 \right) \cdot (p_5 + p_6 z_4^n + p_7 z_4^n z_5^n) \right].$$

(2.25)

Substituting in the high energy limit values of the momentum scalar products gives the eight-tachyon amplitude

$$A_8 \simeq \int_0^1 \prod_{i=1}^5 dz_i f(z_1, z_2, z_4, z_5) z_3^{-1-\alpha(t_3)}$$

$$\times \exp \left[ \sum_{n=1}^{\infty} \frac{z_n^3}{n} \left( z_1^n z_2^n (s_{25} - s_{35}) + z_1^n z_2^n z_4^n (s_{26} - s_{25} + s_{35} - s_{36}) + s_{45} + z_1^n z_2^n z_4^n z_5^n \hat{s} + z_2^n (s_{35} - s_{45}) + z_4^n (s_{46} - s_{45}) + z_4^n z_5^n (s_{36} - s_{35} + s_{45} - s_{46}) + z_2^n z_4^n z_5^n (s_{37} - s_{36} + s_{46} - s_{47}) \right) \right].$$

(2.26)

where, now,

$$\hat{s} = (s_{81} - s_{26} + s_{36} - s_{37}).$$

(2.27)

Applying the four-dimensionality constraints

$$\frac{s_{81}s_{36}}{s_{26}s_{37}} = 1, \quad \frac{s_{81}s_{35}}{s_{25}s_{37}} = 1, \quad \frac{s_{81}s_{46}}{s_{26}s_{47}} = 1, \quad \frac{s_{81}s_{45}}{s_{25}s_{47}} = 1,$$

(2.28)

factorizes the argument of the exponential yielding

$$A_8 \simeq \int_0^1 \prod_{i=1}^5 dz_i f(z_1, z_2, z_4, z_5) z_3^{-1-\alpha(t_3)}$$

$$\times \exp \left[ \hat{s} \sum_{n=1}^{\infty} \frac{z_n^3}{n} (z_1^n - x_1)(z_2^n - x_2)(z_4^n - x_4)(z_5^n - x_5) \right].$$

(2.29)
where
\[ \begin{align*}
x_1 &= \frac{s_{36} - s_{37} + s_{47} - s_{46}}{s}, \\
x_2 &= \frac{s_{47} - s_{46}}{s_{37} - s_{36} + s_{46} - s_{47}}, \\
x_4 &= \frac{s_{25} - s_{35}}{s_{26} - s_{25} + s_{35} - s_{36}}, \\
x_5 &= \frac{s_{25} - s_{26} + s_{36} - s_{35}}{s}.
\end{align*} \tag{2.30} \]

For the critical point \((x_1, x_2, x_4, x_5)\) to be inside the integration region, we must place twist on each of the associated propagators, and apply an additional four-dimensionality constraint:
\[ \frac{s_{36}s_{45}}{s_{35}s_{46}} = 1. \tag{2.31} \]

Consequently, due to the twists we have the sign changes
\[ s_{26}, s_{35}, s_{37}, s_{46} \rightarrow -\infty e^{i\delta}. \tag{2.32} \]

To remove the first two powers of \(z_3\) in the exponential in \((2.29)\), and to obtain a leading trajectory, we will integrate around
\[ |z_1 - x_1| \leq \epsilon, \quad |z_2 - x_2| \leq \epsilon, \quad |z_4 - \sqrt{x_4}| \leq \epsilon, \quad |z_5 - \sqrt{x_5}| \leq \epsilon. \tag{2.33} \]

Clearly, this is just one of many critical points that we could have chosen. By writing \(z_4^2 - x_4 = (z_4 - \sqrt{x_4})(z_4 + \sqrt{x_4}) \sim 2\sqrt{x_4}(z_4 - \sqrt{x_4}), \) etc., and shifting the \(z's\), we find
\[ A_8 \sim f(x_1, x_2, \sqrt{x_4}, \sqrt{x_5}) \int_0^1 dz_3 z_3^{-1-\alpha(t_3)} e^{\tilde{z}_3 z_3^3 c} \times \int_{-\epsilon}^{\epsilon} dz_1 \int_{-\epsilon}^{\epsilon} dz_2 \exp \left[ z_3 z_1 z_2 \tilde{s}(\sqrt{x_4} - x_4)(\sqrt{x_5} - x_5) \right] \tag{2.34} \]
\[ \times \int_{-\epsilon}^{\epsilon} dz_4 \int_{-\epsilon}^{\epsilon} dz_5 \exp[2z_3^2 z_4 z_5 \tilde{s}\sqrt{x_4 x_5}(x_1^2 - x_1)(x_2^2 - x_2)], \]

where
\[ c = \frac{1}{3}(x_1^3 - x_1)(x_2^3 - x_2)(x_3^3 - x_4)(x_4^3 - x_5). \tag{2.35} \]

The last four integrals in \((2.34)\) can be done in pairs, resulting in
\[ A_8 \sim -4\pi^2 \left( 2z_3^2 x_4 x_5(x_1^2 - x_1)(x_2^2 - x_2)(1 - \sqrt{x_4})(1 - \sqrt{x_5}) \right)^{-1} \times f(x_1, x_2, \sqrt{x_4}, \sqrt{x_5}) \int_0^1 dz_3 z_3^{-4-\alpha(t_3)} e^{\tilde{z}_3 z_3^3 c}. \tag{2.36} \]
Using (2.24) then gives
\[
A_8 \sim -\frac{4}{3} f(x_1, x_2, \sqrt{x_4}, \sqrt{x_5})^2 \left((-\tilde{s} c)^{\frac{1}{3} \alpha(t_3)} + 1 \Gamma(-\frac{1}{3} \alpha(t_3) - 1) \right)
\times \left(2\tilde{s}^2 x_4 x_5 (x_1^2 - x_1)(x_2^2 - x_2)(1 - \sqrt{x_4})(1 - \sqrt{x_5}) \right)^{-1}.
\]

(2.37)

Again, we analytically continue back by replacing $-\tilde{s}$ with $e^{-i\pi\tilde{s}}$. Thus, we find the Regge behavior $A_8 \propto \tilde{s}^{\frac{1}{3} \alpha(t_3) - 1} \equiv \tilde{s}^\gamma(t_3)$, which corresponds to the second sister trajectory. The first pole at $\gamma(t_3) = -2$ cancels the pole of the first sister trajectory at $\beta(t_3) = -2$. The daughters of $\gamma(t_3)$ cancel the other poles at $\beta(t_3) = -3, -4, \ldots$.

When the central propagator in Fig. 3 carries the second sister $\gamma(t)$, the adjacent propagators, $z_2$ and $z_4$, see the first sister $\beta(t)$. Each of these sisters can easily be computed by constructing the exponential term in (2.25) from the appropriate overlap quantities, taking the corresponding high energy limit and then integrating over a 2-tuple. Finally, if we had chosen to initially integrate over a 2-tuple critical point for $z_3$, then we would have found the first sister trajectory, $\beta(t)$. 
CHAPTER 3  
DOUBLE-LOOP FOUR-TACHYON SCATTERING

We now adopt our procedure to handle loop corrections. As in the tree level case, we must first isolate terms in the corresponding amplitude which overlap the appropriate propagator. In particular, to search for sisters in the double-loop four-point amplitude, we consider the limiting situation where the two loops are sufficiently separated such that they and the connecting propagator can be treated as individual objects. Two such topologies are shown in Fig. 4. Both may be constructed by sewing together two single-loop diagrams. For this, we use the formalism from appendix D of Di Vecchia et al. 23 where the open string N-point multi-loop vertex has the form

\[ V_{(N;g)} \propto \int d^D \hat{p} \exp \left[ i\tau \cdot \hat{p} + \hat{p} \cdot B + C \right]. \]  

(3.1)

and where \( \tau \) is the period matrix. Completing the square and integrating over the loop momentum \( \hat{p} \) gives

\[ V_{(N;g)} \propto \frac{1}{(\det i\pi \tau)^{D/2}} \exp \left[ -i\pi B \cdot \tau^{-1} \cdot B + C \right]. \]  

(3.2)

The factorized four-tachyon double-loop amplitude is then

\[ A_{(4;2)} = \int_0^1 dz \prod_{i=1}^4 dz_i \int d\mu(0,p_1,p_2) \left| \exp \left[ -\frac{1}{2} \frac{B^2_L}{\ln k_1} + C_L \right] \right| \times z^{L_{0}^{(c)} - 2} \exp \left[ -\frac{1}{2} \frac{(B^R_{c})^2}{\ln k_2} + C^R_{c} \right] \left| 0, p_3, p_4 \right), \]  

(3.3)

where the subscripts \( L \) and \( R \) refer to the left and right loop, resp., and the superscript on \( L_{0}^{(c)} \) labels the leg connecting the loops. The period matrix
Figure 4. Two distinct topologies for producing the $\beta(t)$ sister in double-loop four-point scattering.

has been reduced to the single-loop case $\tau = 2\pi i \ln k$, where $k$ will be defined below. The details of the measure $d\mu$, which is a function of $k_1$ and $k_2$, may be suppressed in the analysis below as long as we avoid the boundaries of the integration region.

In the multi-loop case the coefficient $B_\mu$ in (3.1) is given by (with $\alpha' = 1$)

$$B_\mu = \sqrt{2} \sum_{i=1}^{3} \sum_{m=0}^{\infty} \frac{\alpha_m^{(i)}}{m!} \partial_z^m \left( (\mu) \sum_{\alpha} \ln \frac{\xi_\alpha - T_\alpha(V_i(z))}{\eta_\alpha - T_\alpha(V_i(z))} \right) z=0,$$  

(3.4)

where $z_0, \eta_\mu$, and $\xi_\mu$ are fixed points, and a product of Schottky group elements is defined by

$$T_\alpha = S_{\mu_1}^{n_1} S_{\mu_2}^{n_2} \cdots S_{\mu_r}^{n_r} \quad r = 1, 2, \ldots, g; \quad n_i \in Z/\{0\}; \quad \mu_i \neq \mu_{i+1},$$  

(3.5)

where $g$ is the genus number. Also, $^{(\mu)} \sum_{\alpha}$ means that the sum is over all elements of the Schottky group except that the leftmost element in $T_\alpha$ can not be $S_\mu^n$. In the single-loop case $T_\alpha = S_{1}^{n_1}$ and $S_{1}^{n_1}(y) = k^n y$, where $k$ is the multiplier and related to the radius of the loop. Here, however, the sum restriction leaves just the identity. Finally, for one loop $\xi_1 \rightarrow \infty, \eta_1 \rightarrow 0$. Thus, dropping the loop index,

$$B = \sqrt{2} \sum_{i=1}^{3} \sum_{m=0}^{\infty} \frac{\alpha_m^{(i)}}{m!} \left( \partial_z^m \ln \frac{z_0}{V_i(z)} \right) z=0.$$  

(3.6)
where the projective transformation is explicitly given by

\[ V_i(z) = \frac{z_{i-1}(z_i - z_{i+1})z + z_i(z_{i+1} - z_{i-1})}{(z_i - z_{i+1})z + (z_{i+1} - z_{i-1})} \equiv \frac{a_1z + a_2}{a_3z + a_4}. \quad (3.7) \]

To reduce (3.3) we will need the commutator

\[ [B, \alpha_m^{(c)}] = -\sqrt{2} \sum_{m=1}^{\infty} \frac{1}{(m-1)!} \left( \partial^n_z \ln V_c(z) \right)_{z=0}. \quad (3.8) \]

Partial derivatives of the projective transformation can easily be taken giving

\[ \partial_z V_i(z = 0) = \frac{a_1a_4 - a_2a_3}{(a_3z + a_4)^2} \bigg|_{z=0} = \frac{(z_i - z_{i+1})(z_{i-1} - z_i)}{(z_{i+1} - z_{i-1})}. \quad (3.9) \]

or, more generally,

\[ \partial^m_z V_i(z = 0) = \frac{m!(-a_3)^{m-1}(a_1a_4 - a_2a_3)}{(a_3z + a_4)^{m+1}} \bigg|_{z=0} = (-)^{m-1}m! \frac{(z_i - z_{i+1})^m(z_{i-1} - z_i)}{(z_{i+1} - z_{i-1})^m}. \quad (3.10) \]

The single-loop three-point diagram is constructed by sewing together two legs of a five-point diagram, and then fixing three of the projective coordinates. For that case, following Di Vecchia et al. \(^4\) we sew together legs 3 and 4 and then choose \( z_3 = k, z_4 = \infty, \) and \( z_5 = 1. \) In the present case, we will associate the coordinate \( z_5 \) with the connecting leg coordinate \( z_c. \) This gives \( \partial_z V_c(z = 0) = z_1 - 1, \) along with \( \partial^m_z V_c(z = 0) = 0 \) for \( m \geq 2. \) Thus, the commutator (3.8) becomes

\[ [B, \alpha_m^{(c)}] = -\sqrt{2} \sum_{m=1}^{\infty} \frac{(-)^{m+1}(V_c'(z))^m}{V_c(z)^m} \bigg|_{z=0} = \sqrt{2} \sum_{m=1}^{\infty} (1 - z_1)^m. \quad (3.11) \]

Next, the coefficient \( C \) in (3.1) is given by

\[
C = \sum_{i=1}^{3} \sum_{m=0}^{\infty} \alpha_{(i)} \cdot \frac{1}{m!} \partial^m_z \ln[V_i'(z)]_{z=0} \\
+ 2 \sum_{i,j=1}^{3} \sum_{i<j} \frac{\alpha_{(i)} \cdot \alpha_{(j)}}{n! \cdot m!} \partial^y_z \partial^m_z \ln[V_i(y) - V_j(z)]_{y=z=0} \\
+ \sum_{i,j=1}^{3} \sum_{i<j} \frac{\alpha_{(i)} \cdot \alpha_{(j)}}{n! \cdot m!} \partial^y_z \partial^m_z \ln \frac{E(V_i(y), V_j(z))}{V_i(y) - V_j(z)} \bigg|_{y=z=0}.
\]
where the prime form is defined by

$$E(z, w) = (z - w) \prod_{\alpha} \frac{z - T_\alpha(w)}{z - T_\alpha(z)} \frac{w - T_\alpha(z)}{w - T_\alpha(w)}, \quad (3.13)$$

and the $'$ indicates the identity is not included. For a single loop the prime form reduces to

$$E(z, w) = (z - w) \prod_{n=1}^{\infty} \frac{z - k^n w}{z - k^n z} \frac{w - k^n z}{w - k^n w}. \quad (3.14)$$

Below, we will need the commutator

$$\langle 0 | [C, a_m^{\dagger(c)}] | 0, p_i \rangle = \sum_{m=1}^{\infty} \frac{p}{(m-1)!} \partial_z^m \ln[V_c'(z)] \bigg|_{z=0}$$

$$+ 2 \sum_{i \neq c} \sum_{m=1}^{\infty} \frac{p_i}{(m-1)!} \partial_z^m \ln[z_i - V_c(z)] \bigg|_{z=0}$$

$$+ 2 \sum_{i} \sum_{m=1}^{\infty} \frac{p_i}{(m-1)!} \partial_z^m \ln \prod_{n=1}^{\infty} \frac{z_i - k^n V_c(z)}{z_i - k^n z_i} \frac{V_c(z) - k^n z_i}{V_c(z) - k^n V_c(z)} \bigg|_{z=0}, \quad (3.15)$$

where $p_c = p$. Due to momentum conservation we can neglect the second denominator in the last term. Further, in the high energy limit $s \to \infty$, we have $p_1 \cdot p_3 \to s/2$, $p_1 \cdot p_4 \to -s/2$, $p_2 \cdot p_3 \to -s/2$ and $p_2 \cdot p_4 \to s/2$. These imply,

$$p \cdot p_i = (p_1 + p_2) \cdot p_i \to 0. \quad (3.16)$$

Consequently, some of the terms in (3.15) do not survive the high energy limit in (3.3). This permits us to drop the entire first term, and the $i = c$ term in the last sum. Rearrangement then yields

$$\langle 0 | [C, a_m^{\dagger(c)}] | 0, p_i \rangle = 2 \sum_{i \neq c} \sum_{m=1}^{\infty} \frac{p_i}{(m-1)!} \partial_z^m \ln \prod_{n=0}^{\infty} [z_i - k^n V_c(z)] \prod_{r=1}^{\infty} [V_c(z) - k^r z_i] \bigg|_{z=0}. \quad (3.17)$$
Taking the derivatives gives

\[ \langle 0 | [C, \alpha_m^{(c)}] | 0, p_i \rangle = 2 \sum_{i \neq c} p_i \left[ \sum_{m=1}^{\infty} \frac{-k_{mn}(V_c'(z))^m}{(z_i - k_{n}V_c(z))^m} \right. \\
+ \left. \sum_{m,n=1}^{\infty} \frac{(-1)^{m+1}(V_c'(z))^m}{(V_c(z) - k_{n}z_i)^m} \right] z=0. \]  

(3.18)

which simplifies to

\[ \langle 0 | [C, \alpha_m^{(c)}] | 0, p_i \rangle = -2 \sum_{i \neq c} p_i \left[ \sum_{m=1}^{\infty} \frac{k_{mn}(1 - z_1)^m}{(k_{n} - z_i)^m} \right. \\
+ \left. \sum_{m,n=1}^{\infty} \frac{(1 - z_1)^m}{(1 - k_{n}z_i)^m} \right]. \]  

(3.19)

We also need the single-loop result

\[ \exp \left[ -\frac{1}{2 \ln k_1} \right] | 0, p_1, p_2 \rangle = \psi_{12}^{p_1, p_2} | 0, p_1, p_2 \rangle, \]  

(3.20)

where \( \psi \) arises in planar loop amplitudes and can be expressed in terms of the Jacobi theta function. Substituting (3.11) and (3.19) into (3.3) then gives

\[ A_{4;2} \approx \int_0^1 dz z^{-1 - \alpha(t)} \int d\mu \int_0^1 d\mu' \prod_{i=1}^{4} dz_i \psi_{12}^{2p_1, p_2} \psi_{34}^{2p_3, p_4} \times \exp \left[ \sum_{m=1}^{\infty} \frac{z^m(1 - z_1)^m(1 - z_3)^m}{m} \left( \sum_{i,j} \frac{p_i \cdot p_j \ln z_i \ln z_j}{\ln k_1 \ln k_2} \right) \right. \\
- \sum_{i \neq c} \sum_{j \neq c} p_i \cdot p_j \frac{\ln z_i}{\ln k_1} \left[ \sum_{n=0}^{\infty} \frac{k_{2n}^{mn}}{(k_{2n}^n - z_j)^m} \right. \\
+ \left. \sum_{n=1}^{\infty} \frac{1}{(1 - k_{2n}^n z_j)^m} \right] \\
- \sum_{i \neq c} \sum_{j} p_i \cdot p_j \frac{\ln z_j}{\ln k_2} \left[ \sum_{n=0}^{\infty} \frac{k_{1n}^{mn}}{(k_{1n}^n - z_i)^m} \right. \\
+ \left. \sum_{n=1}^{\infty} \frac{1}{(1 - k_{1n}^n z_i)^m} \right] \\
+ \sum_{i \neq c} \sum_{j \neq c} p_i \cdot p_j \left( \sum_{n=0}^{\infty} \frac{k_{1n}^{mn}}{(k_{1n}^n - z_i)^m} \right. \\
+ \left. \sum_{n=1}^{\infty} \frac{1}{(1 - k_{1n}^n z_i)^m} \right) \\
\times \left( \sum_{r=0}^{\infty} \frac{k_{2r}^{mr}}{(k_{2r}^r - z_j)^m} \right. \\
+ \left. \sum_{r=1}^{\infty} \frac{1}{(1 - k_{2r}^r z_j)^m} \right). \]  

(3.21)

where \( i \) and \( j \) correspond to the different loops, and we have dropped a momentum independent factor which can be ignored in the high energy limit.
Replacing the momentum scalar products by their high energy limits allows us to factorize the argument of the exponential to get

$$A_{(4;2)} \approx \int_0^1 dz z^{-1 - \alpha(t)} \int d\mu \prod_{i=1}^4 dz_i (\psi_{12} \psi_{34})^{-1 - \alpha(t)}$$

$$\times \exp \left[ \sum_{m=1}^{\infty} \frac{z^m (1 - z_1)^m (1 - z_3)^m}{m} g_m(z_1, z_2, k_1) g_m(z_3, z_4, k_2) \right],$$

(3.22)

where we have defined

$$g_m(x, y, k) = \frac{\ln x}{\ln k} - \sum_{n=0}^{\infty} \frac{k^{mn}}{(k^n - x)^m} - \sum_{n=1}^{\infty} \frac{1}{(1 - k^n x)^m} < x \leftrightarrow y >.$$  (3.23)

The function $g_m(x, y, k)$ is for orientable planar loops and is essentially the $m$th derivative of $\ln \psi$. Thus, we can immediately write down the expression in the non-orientable case:

$$g_m^no(x, y, k) = \frac{\ln x}{\ln k} - \sum_{n=0}^{\infty} \frac{(-k)^{mn}}{((-k)^n - x)^m} - \sum_{n=1}^{\infty} \frac{1}{(1 - (-k)^n x)^m} < x \leftrightarrow y >,$$

(3.24)

and for the non-planar case:

$$g_m^np(x, y, k) = \frac{\ln x}{\ln k} - \sum_{n=0}^{\infty} \frac{(-1)^m k^{nm}}{((-1)^n k^n - x)^m} - \sum_{n=1}^{\infty} \frac{1}{(1 + k^n x)^m} < x \leftrightarrow y >.$$  (3.25)

Now, we search for critical points which do not reside on the boundary of the integration region. Unfortunately, due to its complicated form, one must numerically search for zeros in $g_m(x, y, k)$. It is found that $g_m(x, y, k)$, for all $m$, does indeed possess zeros that are exclusively within the integration range. These zeros generate the critical-point curve $x = P(y, k)$, for some function $P(y, k)$ which satisfies $g_m(P(y, k), y, k) = 0$. In addition, numerically analysis indicates that both non-orientable and non-planar cases also possess critical-point curves. In all these cases the zeros do not seem to be confined to any particular region of integration space.
This case differs from the tree calculation in two respects. First, to factorize Eq. (3.21) it was not necessary to impose a dimensionality constraint. Clearly, this is due to the fact that there are only four interacting particles, and not due to the loops. Second, unlike the tree amplitudes, the presence of twists is not significant. In the former case, the twists were necessary to change the sign of some of the energies to place critical points inside the integration region. In the loop amplitudes, the signs change as a result of the periodicity of the Jacobi theta function.

Continuing with the calculation, in the limit $s \to \infty e^{i\theta}$ (3.22) becomes

$$A_{(4;2)} \sim \int_0^1 dz \int_0^1 d\mu \int_0^1 dz_i \psi_{12} \psi_{34}^{-1} e^{s z^2 h_2} \left(1 - z_1 \right) \left(1 - z_3 \right) g_1(z_1, z_2, k_1) g_1(z_3, z_4, k_2) \exp \left[ s z_1 z_3 h_1 \right],$$

where

$$h_2 = \frac{1}{2} (1 - z_1)^2 (1 - z_3)^2 g_2(z_1, z_2, k_1) g_2(z_3, z_4, k_2)$$

(3.27)

We will evaluate about the critical curve

$$| z_1 - P(z_2, k_1) | \leq \epsilon, \quad | z_3 - P(z_4, k_2) | \leq \epsilon.$$  

(3.28)

Expanding the $g_1$'s about this curve, and then shifting $z_1$ and $z_3$, gives

$$A_{(4;2)} \sim \int_0^1 dz \int_0^1 d\mu \int_0^1 dz_2 dz_4 \psi_{12} \psi_{34}^{-1} e^{s z^2 h_2} \left(1 - P(z_2, k_1) \right) \left(1 - P(z_4, k_2) \right) g_1(z_3 - P(z_4, k_2), z_4, k_2).$$

(3.29)

and $h_2$, $\psi_{12}$, and $\psi_{34}$ are now evaluated on the critical curve. The integration of $z_1$ and $z_3$ proceeds as before, giving

$$A_{(4;2)} \sim i \pi s^{-1} \int_0^1 dz \int_0^1 d\mu \int_0^1 dz_2 dz_4 \psi_{12} \psi_{34}^{-1} e^{s z^2 h_2} \times \left(1 - P(z_2, k_1) \right) \left(1 - P(z_4, k_2) \right).$$

(3.31)
Similarly, the $z$ integration is also easily done giving

$$A_{(4;2)} \sim -i\pi e^{-i\pi \beta(t)} s^{\beta(t)} \Gamma(-\beta(t) - 1) \int d\mu$$

$$\times \int_0^1 dz_2 dz_4 (\psi_{12} \psi_{34})^{-1 - \alpha(t)} h_1^{-1} h_2^{\beta(t)+1},$$

which exhibits the first sister trajectory $\beta(t)$. Since the integrands involve derivatives of the Jacobi theta functions, we are unable to complete the calculation showing explicitly that the sister does not decouple. For the planar diagram, however, in the special case $\beta(t) = -1$, it can easily be shown that the signs of each of the integrand factors are the same over the entire integration region. On the other hand, to show that decoupling does not occur in the non-orientable and non-planar cases is more difficult, although the results of the next chapter indicate that the sister survives the latter case.

The existence of the second sister requires that two of the $g$'s share the same critical point. Using

$$(1 + x)^{-(r+1)} - (1 + x)^{-r} = -xe^x,$$  \hspace{1cm} (3.33)

it follows that

$$g_r(x, y, k) - g_{r+1}(x, y, k) = \sum_{n=1}^{\infty} \left( k^n x e^{-k^n x} - k^n y e^{-k^n y} \right)$$

$$+ \sum_{n=0}^{\infty} \left( \frac{x}{k^n} e^{-x/k^n} - \frac{y}{k^n} e^{-y/k^n} \right).$$

Since the difference is independent of the index $r$, for any given critical point either one $g_r$ vanishes, resulting in a single sister, or they all vanish simultaneously. In the latter case, (3.22) results in the form

$$A_{(4;2)} \sim \int_0^1 dz z^{-1 - \alpha(t)} \int d\mu \int_0^1 dz_2 dz_4 (\psi_{12} \psi_{34})^{-1 - \alpha(t)}$$

$$\times \int_{-\epsilon}^{\epsilon} dz_1 dz_3 \exp \left[ s z_1 z_3 \sum_{m=1}^{\infty} z^m e_m \right].$$  \hspace{1cm} (3.35)
Integrating over $z_1$ and $z_3$, we obtain

$$A_{(4;2)} \approx i \pi s^{-1} \int_0^1 dz z^{-1-\alpha(t)} \int d\mu \times \int_0^1 dz_2 dz_4 (\psi_{12} \psi_{34})^{-1-\alpha(t)} \left( \sum_{m=1}^{\infty} z^m e_m \right)^{-1}.$$  

(3.36)

The right factor gives a $z^{-1}$ in leading order. Consequently, the $z$ integral generates a leading pole at $\alpha(t) = -1$, whereas the second sister requires $\alpha(t) = -2$.

Presumably, the $\gamma(t)$ trajectory is present if there are at least two loops on both sides of the propagator. We suspect that, in this case, there would be a factorization of the form

$$G_m(x, y, k_1, k_2) = g_m(x, y, k_1)g_m(x, y, k_2)$$  

(3.37)

where $k_1$ and $k_2$ correspond to same-side loops. In Fig. 5, we display two distinct possible multi-loop topologies for producing the higher order sisters.
Figure 6. The Regge cut behavior is across the dotted lines.

In both cases the central propagator may allow up to the $m^{th}$ sister if there are at least $m$ loops on either side. However, evaluating Fig. 5a is not practical since the Schottky representation of the prime form (3.13) is much too formal when two or more unfactorized loops are present. On the other hand, since Fig. 5b completely factorizes the loops it requires no more than the techniques presented in this chapter.

The sister trajectories may also appear across propagators which are embedded in an irreducible diagram. An example is the double-loop diagram displayed in Fig. 6. The sister here may be across one of the horizontal propagators. Such diagrams are, however, dominated by the behavior of Regge cuts. In the present case, the cut in Fig. 6a gives

$$A \sim \frac{e^{\frac{\alpha(t)}{2} \ln s}}{(\ln s)^p},$$

for some $p$ at fixed $t$. The cut has the same Regge slope as the first sister, yet its $\alpha(t)$-intercept is higher. In general, the $n^{th}$ cut occurs at the same order as that of the $n^{th}$ sister, but with a trajectory lying above the sister. This implies that the collective behavior of the cuts would actually exceed the
CM bound. A high energy analysis of the entire moduli space, such as that of Gross and Mende, would be dominated by the cuts. This is supported, in part, by their proposal that the fixed $t$ behavior have the form

$$A \sim \frac{\alpha(s) \ln s}{e^{s+1} (\ln s)^{12g}}. \tag{3.39}$$

where $g$ is the genus number. The single-loop amplitude, computed first in the fixed angle limit, was shown explicitly to reduce to (3.39), for $p = 1$, in the fixed $t$ limit $\theta \to 0$. 
CHAPTER 4
OPEN STRING SISTERS IN
CLOSED STRING SCATTERING

An unexpected result of the last chapter is uncovered by considering the non-planar diagram in Fig. 7. The central propagator that carries the sister $\beta(t)$ is that of the open string, while the non-planar loops on either side contain closed string poles. This raises the interesting possibility of open string sisters coupling to closed string propagators as in the diagram shown in Fig. 8. Below we show that this is in fact the case. In the case of the Heterotic string, however, the diagram in Fig. 8 decouples since the open string propagator can not accommodate the achiral boundary conditions required by the closed string propagators.

The amplitude for four-tachyon closed string scattering with an intermediate open string propagator, takes the form

$$A_G^2 = \left(\frac{1}{4\pi}\right)^2 \int_{|z_1| \leq 1} d^2z_1d^2z_{2\text{cl}}\langle 0, p_4 | V^\dagger(p_3, z_1^{-1}, \bar{z}_1^{-1}) \times_o(0 | \Upsilon(A^\dagger, a) | 0)_{\text{cl}}$$
$$\times \Delta_o \times_{\text{cl}}(0 | \Upsilon(A, a^\dagger) | 0)_{o} \times V(p_2, z_2, \bar{z}_2) | 0, p_1)_{\text{cl}}.$$  \hspace{1cm} (4.1)

Among the many expressions appearing in the literature for the transition operator $\Upsilon$ between the open and closed string state, we will use that of Shapiro and Thorn.\textsuperscript{25} We will ignore here the ghosts terms given in their explicit expression for $\Upsilon$. These give a non-trivial contribution only if loops are present. Even then, the ghosts can be ignored since they have no bearing on the calculation which focuses on the exponential contributions away from the integration
Figure 7. Non-planar double-loop four-point diagram. The loops contain closed string poles.

Figure 8. Four-point closed string interaction with an intermediate open string propagator.

boundary region. The transition operator is then given by

\[ \Upsilon(A, a^\dagger) = \exp \left[ -\sqrt{2} \sum_{n,m=0}^{\infty} C_{nm}^{(1)} A_n^s \cdot \alpha_{-2m-1} + \sum_{n,m=0}^{\infty} C_{nm}^{(2)} \alpha_{-2n-1} \cdot \alpha_{-2m-1} \right. \]

\[ \left. - \frac{1}{2} \sum_{(n,m)\neq(0,0)}^{\infty} C_{nm}^{(3)} A_n^s \cdot A_m^s - \frac{1}{\sqrt{2}} \sum_{m=1}^{\infty} \frac{1}{m} A_m^s \cdot \alpha_{-2m} \right], \]

where,

\[ C_{nm}^{(1)} = \frac{(-)^{n+m}}{2m-2n+1} \left[ \begin{array}{c} -\frac{1}{2} \\ n \end{array} \right] \left[ \begin{array}{c} -\frac{1}{2} \\ m \end{array} \right], \] (4.3)

\[ C_{nm}^{(2)} = \frac{1}{4n+m+1} \left[ \begin{array}{c} -\frac{1}{2} \\ n \end{array} \right] \left[ \begin{array}{c} -\frac{1}{2} \\ m \end{array} \right], \] (4.4)

and

\[ C_{nm}^{(3)} = \frac{(-)^{n+m}}{n+m} \left[ \begin{array}{c} -\frac{1}{2} \\ n \end{array} \right] \left[ \begin{array}{c} -\frac{1}{2} \\ m \end{array} \right]. \] (4.5)
The sine and cosine oscillators of the closed string are given by

\[ A^c_r = \frac{1}{\sqrt{2}} (A_r + \tilde{A}_r), \quad A^s_r = \frac{i}{\sqrt{2}} (A_r - \tilde{A}_r), \]  

(4.6)

where \( \tilde{A}_r \) and \( A_r \) correspond to the left and right movers, respectively. In terms of the sine and cosine modes, the closed string vertex operator is written

\[
V(p_2, z, \bar{z}) \mid 0, p_1 \rangle_{cl} = e^{i(p_1 + p_2) \cdot x} \mid z \mid^{-2 - \alpha(t)} \exp \left[ p_2 \cdot \sum_{n=1}^{\infty} \left( \frac{A^c_n}{n} (z^n + \bar{z}^n) - i \frac{A^s_n}{n} (z^n - \bar{z}^n) \right) \right] \mid 0 \rangle_{cl},
\]

(4.7)

where the first two factors are the zero modes.

In Eq. (4.7) we have written the closed string trajectory \( \alpha(t) \equiv \alpha_{cl}(t) \). To eliminate confusing notation, we will write the open string trajectory also in terms of \( \alpha(t) \), i.e., \( \alpha_o(t) = 2\alpha(t) - 3 \). Then, we have

\[ z^{-t-2} \mid 0 \rangle_o = z^{2-2\alpha(t)} \mid 0 \rangle_o = z^{2-2\alpha(t)} \mid 0 \rangle_o. \]

(4.8)

Pushing \( \Upsilon \) to the right, \( \Upsilon \dagger \) to the left, moving the propagator to the right, and then using momentum conservation to eliminate part of the zero modes, Eq.
(4.2) becomes
\[
A_{G^2} = \left( \frac{1}{4\pi} \right)^2 \int_0^1 \int_{|z_i| \leq 1} dz_1 \frac{d^2z_1 d^2z_2}{|z_1 z_2|^{2\alpha(t)}} \times \exp \left[ -\frac{1}{\sqrt{2}} p_3 \cdot \sum_{n=1}^\infty \left( -\frac{1}{n} \alpha_{2n} (z_1^n + \bar{z}_1^n) + i \sum_{m=0}^\infty C_{nm}^{(1)} \alpha_{2m+1} (z_1^n - \bar{z}_1^n) \right) \right] \times \exp \left[ \sum_{n,m=0}^\infty C_{nm}^{(2)} \alpha_{2n+1} \cdot \alpha_{2m+1} \right] \times \exp \left[ \sum_{n,m=0}^\infty C_{nm}^{(2)} \alpha_{-2n-1} \cdot \alpha_{-2m-1} z^{2(n+m+1)} \right] \times \exp \left[ \sum_{n=1}^\infty \left( -\frac{1}{n} \alpha_{-2n} z^{2n}(z_2^n + \bar{z}_2^n) + i \sum_{m=0}^\infty C_{nm}^{(1)} \alpha_{-2m-1} z^{2m+1}(z_2^n - \bar{z}_2^n) \right) \right] | 0\rangle_0.
\]

We can easily move the even oscillators through to the vacuum states since they only appear at the far left and far right. This produces the factor
\[
\exp \left[ -p_2 \cdot p_3 \sum_{n=1}^\infty \frac{1}{n} z^{2n}(z_1^n + \bar{z}_1^n)(z_2^n + \bar{z}_2^n) \right] = \left| (1 - z^2 z_1 z_2)(1 - z^2 z_1 \bar{z}_2) \right|^{2p_2 p_3},
\]
which can be pulled outside the vacuum states.

Next, pushing the quadratic terms past each other produces the factor
\[
\exp \left[ \sum_{k,n,m=0}^\infty 4(2k+1)C_{nk}^{(2)} C_{km}^{(2)} \alpha_{-2n-1} \cdot \alpha_{2m+1} z^{2(n+k+1)} \right],
\]
and an oscillator independent exponential which we can neglect since it will not survive in the high energy limit. Moving the quadratic terms to the vacuum states will produce no other permanent effect as they pass by the vertex.
operators. Pushing the factor (4.11) past the right vertex then results in

\[ A_G^2 \simeq \left( \frac{1}{4\pi} \right)^2 \int_0^1 dz_1 z^{2-2\alpha(t)} \int_{|z_1| \leq 1} d^2 z_1 d^2 z_2 \ | z_1 z_2 |^{-2-\alpha(t)} \times \left| (1 - z^2 z_1 z_2)(1 - z^2 z_1 \bar{z}_2) \right|^{2p_2 p_3} \times o(0) \exp \left[ -i \sqrt{2} p_3 \cdot \sum_{n=1}^{\infty} C^{(1)}_{n m} \alpha_{2m+1}(z_1^n - \bar{z}_1^n) \right] \times \exp \left[ i \sqrt{2} p_2 \cdot \sum_{n=1}^{\infty} C^{(1)}_{n m} \alpha_{2m-1} z^{2m+1}(z_2^n - \bar{z}_2^n) \times \left( 1 + 4(2m + 1) \sum_{j,k=0}^{\infty} (2k + 1) \alpha_{-2j-1} C^{(2)}_{j k} C^{(2)}_{k m} z^{2(j+k+1)} \right) \right] | 0 \rangle_o.

(4.12)

Again, the quadratic term will not leave any permanent imprint after moving to the left-hand side. Finally,

\[ A_G^2 \simeq \left( \frac{1}{4\pi} \right)^2 \int_0^1 dz_1 z^{2-2\alpha(t)} \int_{|z_1| \leq 1} d^2 z_1 d^2 z_2 \ | z_1 z_2 |^{-2-\alpha(t)} \times \left| (1 - z^2 z_1 z_2)(1 - z^2 z_1 \bar{z}_2) \right|^{2p_2 p_3} \times \exp \left[ 2p_2 \cdot p_3 \sum_{k,n=1}^{\infty} (2m + 1) z^{2m+1} C^{(1)}_{k m} C^{(1)}_{n m} (z_1^k - \bar{z}_1^k)(z_2^n - \bar{z}_2^n) \right].

(4.13)

To perform the sums in the second exponential, we go to the limit \( s \to \infty e^{i\delta} \) and keep only the term linear in \( z \). The sums can now be done by noting

\[ \left[ \frac{1}{n} \right] = \frac{1}{1 - 2n} \left[ -\frac{1}{n} \right]. \]

(4.14)

Thus,

\[ \sum_{n=1}^{\infty} z^n C^{(1)}_{n 0} = \sum_{n=1}^{\infty} \frac{(-z)^n}{1 - 2n} \left[ -\frac{1}{n} \right] = (1 - z)^{\frac{1}{2}} - 1. \]

(4.15)
Eq. (4.13) then becomes

\[ A_G^2 \sim \left( \frac{1}{4\pi} \right)^2 \int_0^1 dz z^{2-2\alpha(t)} \int_{|z| \leq 1} d^2 z_1 d^2 z_2 \left| z_1 z_2 \right|^{-2-\alpha(t)} \]

\[ \times \left| (1 - z^2 z_1 z_2)(1 - z_2 z_1 \bar{z}_2) \right|^{-s-8} \]

\[ \times \exp \left[ -sz \left( \left(1 - z_1 \right)^{1/2} - (1 - \bar{z}_1) \right) \left( (1 - z_2)^{1/2} - (1 - \bar{z}_2) \right) \right]. \]  

(4.16)

As in the double-loop case, since there are only four interacting particles, the argument of the exponential appears in a factorized form without resorting to a dimensionality constraint.

Examining the second exponential term in (4.16), we see that there is a critical point when \( z_1 = \bar{z}_1 \) or \( z_2 = \bar{z}_2 \). Writing \( z = \rho e^{i\theta} \), implies \( \theta = 0 \) or \( \pi \).

To integrate (4.16) about these points, we return to the Taylor series expansion in (4.13) and (4.10), i.e.,

\[ A_G^2 \sim \int_0^1 dz z^{2-2\alpha(t)} \int_0^1 d\rho_1 d\rho_2 (\rho_1 \rho_2)^{-1-\alpha(t)} \int_0^{2\pi} \frac{d\theta_1}{4\pi} \frac{d\theta_2}{4\pi} \]

\[ \times \exp \left[ 2sz^2 \rho_1 \rho_2 \cos \theta_1 \cos \theta_2 \right] \]

\[ \times \exp \left[ 4sz \sum_{k,n=1}^{\infty} C_{k0}^{(1)} C_{n0}^{(1)} \rho_1^k \rho_2^n \sin(k \theta_1) \sin(n \theta_2) \right]. \]  

(4.17)

We expand by setting \( \sin(n\theta) \approx n\theta \) for \( \theta = 0 \), and \( \sin(n\theta) \approx (-)^n n\theta \) for \( \theta = \pi \).

The \( \rho \) sums for \( \theta = 0 \) can easily be carried out as follows:

\[ \sum_{n=1}^{\infty} \rho^n n C_{n0}^{(1)} = \rho \frac{\partial}{\partial \rho} \sum_{n=1}^{\infty} (-\rho)^n \left[ -\frac{1}{2} \right] = -\frac{1}{2} \rho (1 - \rho)^{-1/2}. \]

(4.18)

With a similar expression for \( \theta = \pi \), eq. (4.17) becomes

\[ A_G^2 \sim 2 \int_0^1 dz z^{2-2\alpha(t)} \int_0^1 d\rho_1 d\rho_2 (\rho_1 \rho_2)^{-1-\alpha(t)} \int_{-\epsilon}^{\epsilon} \frac{d\theta_1}{4\pi} \frac{d\theta_2}{4\pi} \]

\[ \times \left( \exp \left[ sz \theta_1 \theta_2 \rho_1 \rho_2 (1 - \rho_1)^{-1/2} (1 - \rho_2)^{-1/2} + 2s \rho_1 \rho_2 z^2 \right] \right) \]

\[ + \exp \left[ -sz \theta_1 \theta_2 \rho_1 \rho_2 (1 + \rho_1)^{-1/2} (1 + \rho_2)^{-1/2} - 2s \rho_1 \rho_2 z^2 \right] \]  

(4.19)

where the first exponential is for \( \theta_1 \) and \( \theta_2 \) expanded around the same value, and the second for the converse case.
Integration over $\theta_1$ yields

$$A_G^2 \sim -\frac{1}{8s\pi^2} \int_0^1 dz z^{1-2\alpha(t)} \int_0^1 d\rho_1 d\rho_2 (\rho_1 \rho_2)^{-2-\alpha(t)} \int_{y_0}^{y_0} dy \left[ \frac{e^{iy} - e^{-iy}}{y} \right]$$

$$\times \left( (1 - \rho_1)^{1/2} (1 - \rho_2)^{1/2} \exp \left[ 2s \rho_1 \rho_2 z^2 \right] + (1 + \rho_1)^{1/2} (1 + \rho_2)^{1/2} \exp \left[ -2s \rho_1 \rho_2 z^2 \right] \right),$$

(4.20)

where the exact expression for $y_0$ is not needed. The integration over $y$ gives $2i\pi$. Unlike the previous examples, the sister is not necessarily the dominant behavior. This requires that we extend our considerations to higher orders. By Taylor expanding $(1 - \rho)^{1/2}$ and $(1 + \rho)^{1/2}$, we obtain many terms which may indicate the presence of the open string sister, $\beta_o$. To make a firm determination requires some care since the $\beta_o$ trajectory is degenerate with the dilaton trajectories that may appear across the adjacent closed string propagators.

There is no doubt, however, when (4.20) generates a triple pole. The form of the required solution is suggested by the partial wave analysis term

$$\int_C \frac{s^t}{(\alpha_o(t) - t)(\alpha_{cl}(t) - t)(\alpha_{cl}(t) - t)} \rightarrow s^{\alpha(t)} \ln^2 s,$$

(4.21)

for the case $\alpha_o(t) = \alpha_{cl}(t)$. Eq. (4.20) yields this result if we select the $\rho^2$ expansion terms for both $\rho_1$ and $\rho_2$. This gives,

$$A_G^2 \sim -\frac{i}{28s\pi} \int_0^1 dz z^{1-2\alpha(t)} \int_0^1 d\rho_1 d\rho_2 (\rho_1 \rho_2)^{-\alpha(t)}$$

$$\times \left( \exp \left[ 2s \rho_1 \rho_2 z^2 \right] + \exp \left[ -2s \rho_1 \rho_2 z^2 \right] \right).$$

(4.22)

Let $w = \rho_1 \rho_2$. Then

$$A_G^2 \sim -\frac{i}{28s\pi} \int_0^1 dz z^{1-2\alpha(t)} \int_0^1 dw w^{-\alpha(t)} \left( e^{2swz^2} + e^{-2swz^2} \right) \int_w^1 dp_2 p_2^{-1}.$$

(4.23)
The $p_2$ integral easily gives $-\ln w$. Next, defining $w = z^{-2}y$, we integrate over $z$ to obtain

\[
AG^2 \sim \frac{i}{2^{10} s \pi} \int_0^1 dy y^{-\alpha(t)} \ln^2 y \left( e^{2sy} + e^{-2sy} \right)
\]

\[
\times \frac{i}{2^{10} s \pi} \frac{d^2}{d \alpha(t)^2} \int_0^1 dy y^{-\alpha(t)} \left( e^{2sy} + e^{-2sy} \right).
\]

Finally, the end result is

\[
AG^2 \sim i \pi^{-1} \left( 1 - e^{-i \pi \beta_o(t)} \right) 2^{-9+\beta_o(t)} \Gamma(-1 - \beta_o(t)) s^{\beta_o(t)} \ln^2 s,
\]

where the open string sister $\beta_o(t) = \alpha(t) - 2$.

One possible concern that may arise in the above calculation is that in writing Eq. (4.17) we have discarded the term

\[
\left[ (1 - z_1^2 z_2^2)(1 - z_1^2 \bar{z}_1 \bar{z}_2)(1 - z_2^2 \bar{z}_2)(1 - z_2^2 \bar{z}_2) \right]^{m_2^2 + m_3^2}.
\]

When tachyons are present this may diverge at the critical points in the neighborhood $z = \pm | z_1 | = \pm | z_2 | = 1$. Fortunately, the sister trajectory emerges from the other end of the integration region where these quantities approach zero.

In place of Fig. 8b, we could also represent the open string propagator as a disk that is cut out of a plane which parametrizes the world-sheet. In principle, we can recover the situation discussed in this chapter if we impose Neumann boundary conditions on the hole, and then factorize by restricting the locations of the vertices. An alternative case is when the hole obeys Dirichlet boundary conditions. In this case, the open string propagator is physical only when there is zero momentum across it. Since the sister trajectories occur in the limit of large s momentum transfer squared, we can rule out their existence in the Dirichlet theory.
CHAPTER 5
OSCILLATOR REPRESENTATION
OF SISTER TRAJECTORIES

In this chapter we will determine the state representation of the sister trajectories. The basic nature of sister states will differ from the states associated with the standard Regge trajectory $\alpha(t)$ since the corresponding sister poles are not physical. In the space-like $t$ region, the poles have the manifest unphysical characteristic of nonsense, \textit{i.e.}, negative spin $J$. The time-like resonances are not physical either. In exact expressions, the residues associated with physical resonances can always be written as polynomials in the energies.\textsuperscript{10} At tree level, we can see from the final expressions (2.21) and (2.37) that the energies overlapping the sister propagator are not in this form. However, this is not the case in the double-loop expression (3.32) where only a single energy appears. Here, repeating the argument given for the six-point case in Chap. 2, we rely on the fact that the sister has wrong-signature. Constructing the sister states will give us another means for uniformly displaying the unphysical nature of the poles in all regions of $t$.

There exists, yet, a second motivation for being interested in the state representations of the sisters. At about the same time sisters were found in the high energy analysis, they were noticed in an entirely different context by Goldstone\textsuperscript{27} who was investigating the problem of counting physical states at each mass level.\textsuperscript{28,29} This is a nontrivial problem because in D dimensions the physical states transform under the group $O(D - 1)$, whereas, due to gauge invariance, string states fill multiplets of the transverse group $O(D - 2)$. For
the case of four space-time dimensions, the counting problem was solved in 1976 when Goldstone presented the generating function

$$\chi(x, J) = \left[ \sum_{n=1}^{\infty} (1 - x^n)^{-2} \right] \sum_{r=1}^{\infty} x^{rJ + r(r-1)/2} (-1)^{r-1} (1 - x^r)^2. \quad (5.1)$$

This has since been generalized to higher dimensions, and for the Superstring and Heterotic string.\textsuperscript{29,30,31} Expanding out (5.1), the coefficient of $x$ counts the number of $O(3)$ representations of spin $J$, while the exponent is the corresponding mass level. The connection to sisters can be made if in (5.1) one sets the $x$ exponent in the second factor equal to $M^2$, \textit{i.e.},

$$J_r(M) = \frac{M^2}{r} - \frac{1}{2}(r - 1). \quad (5.2)$$

Since poles in the $\alpha(t)$ plane are labeled by $(M^2, J)$, we can identify (5.2) with (2.2).

The state analysis of Goldstone is in the time-like $t$ region, while the high energy analyses exposes dominant behavior in the space-like region. Determining the state representation of the sister trajectories will provide a more direct link between these two approaches. The unifying feature of pole cancellation can be seen in Fig. 9 which, for $\alpha(t) \geq 0$, displays the lowest mass levels obtained from Goldstone’s formula (5.1). The figure shows how the various trajectories conspire to form the physical states(solid dots) and remove some of the pure gauge states(crosses), and that the $\beta(t)$ trajectory in both regimes enters with the opposite sign to the $\alpha(t)$ and $\gamma(t)$ trajectories.

The state representations of the physical states, defined at the poles of the standard Regge $\alpha(t)$ trajectories and its daughters, are well known. The first three states of the leading trajectory displayed in Fig. 9 are given by the tachyon $|0\rangle$, the “photon” $\alpha_{-1} \mid 0\rangle$, and the massive spin two symmetric state $\alpha_{-1}^{[i} \alpha_{-1}^{j]} \mid 0\rangle$, where the transverse index $i = 0, \ldots, D - 2$. Suppressing the space-time index, the general leading state is given by $\alpha_{-1}^n \mid 0\rangle$, $n \geq 0$. It is
important to note that the mode number of the states along the $a(t)$ trajectory differ by one. This implies that by varying $n_r$ in the general open string state

$$a_{-1}^{n_1}a_{-2}^{n_2}\cdots a_{-r}^{n_r}\cdots |0\rangle,$$  

we move along a path in Fig. 9 that parallels the $r^{th}$ sister trajectory. Although the poles of the $r^{th}$ sister trajectory do differ by mode number $r$, the corresponding sister states can not be represented by the physical states (5.3).

Instead, by analogy with the high energy analysis, we must analytically continue away from the states defined by (5.3). To proceed, we will work directly with the factorized scattering amplitude. This isolates the appropriate propagator which allows us project onto it all possible classes of physical states. We are then free to select the states which lead to the sisters. For the six-point
diagram of Fig. 1. again ignoring the twists, we project the physical states onto the central propagator. To preserve unitarity, we insert the corresponding identity operator on adjacent sides of the propagator, i.e.,

\[
A_6 = \int_0^1 \frac{dz_1 dz_3}{z_1 z_3} \langle 0, p_1 | V(p_2, z_1^{-1}) V(p_3, 1) I \frac{1}{L_0 - 1} IV(p_4, 1) V(p_5, z_3) | 0, p_6 \rangle,
\]

(5.4)

where

\[
I = \sum_{n_1=0}^{\infty} \frac{1}{n_1!} \sum_{n_2=0}^{\infty} \frac{1}{2n_2!} \cdots \left( a_{-1}^{n_1} a_{-2}^{n_2} \cdots | 0 \right) \left( 0 | \cdots a_2^{n_2} a_1^{n_1} \right).
\]

(5.5)

The normalizations in (5.5) are fixed by the projector condition \( I^2 = I \), and the commutation relations

\[
[a_\mu^m, a_\nu^n] = m \delta_{m+n} \eta^{\mu \nu}.
\]

(5.6)

When we project \( I \) onto the central propagator, we easily obtain

\[
I \frac{1}{L_0 - 1} I = \prod_{r=1}^{\infty} \sum_{n_r=0}^{\infty} \frac{1}{r^{n_r} n_r!} a_{-r}^{n_r} | 0 \rangle \langle 0 | a_{r}^{n_r} \frac{1}{\sum_{j=1}^{\infty} j n_j - \alpha(t)}.
\]

(5.7)

where \( t \equiv t_2 \). Substituting this into (5.4), using the four-dimensionality constraint (2.9), and then taking the high energy limit \( s_{34} \to \infty \), yields

\[
A_6 \simeq \int_0^1 dz_1 dz_3 z_1^{-1-\alpha(t_1)} z_3^{-1-\alpha(t_3)} (1 - z_1)^{-1 - \alpha(s_{23})} (1 - z_3)^{-1 - \alpha(s_{45})}
\]

\[
\times \prod_{r=1}^{\infty} \sum_{n_r=0}^{\infty} \frac{1}{r^{n_r} n_r!} \sum_{j=1}^{\infty} j n_j - \alpha(t) \left[ \tilde{s} \left( z_1^r - x_1 \right) \left( z_3^r - x_3 \right) \right]^{n_r}.
\]

(5.8)

where \( x_1, x_3 \) and \( \tilde{s} \) are as before. Note, by using the integral representation

\[
\frac{1}{\sum_{j=1}^{\infty} j n_j - \alpha(t)} = \int_0^1 dz \sum_{j=1}^{\infty} j n_j - \alpha(t) - 1,
\]

(5.9)

we can replace the sums in (5.8) by exponential functions to get

\[
A_6 \simeq \int_0^1 dz_1 dz_3 dz_1^{-1-\alpha(t_1)} z_3^{-1-\alpha(t_3)} (1 - z_1)^{-1 - \alpha(s_{23})} (1 - z_3)^{-1 - \alpha(s_{45})}
\]

\[
\times z^{-1-\alpha(t)} \exp \left[ \tilde{s} \sum_{r=1}^{\infty} \frac{z^r}{r} \left( z_1^r - x_1 \right) \left( z_3^r - x_3 \right) \right].
\]

(5.10)
Thus, we have completely recovered Eq. (2.10). In fact, at virtually each step of our computations below, there is a parallel step using the exponentiated form. This provides a useful check on our results, and allows us to be brief in much of the derivation.

As in Chap. 2, to obtain the leading $\beta(t)$ trajectory we require that $(x_1, x_3)$ be a critical point. In this case, not only do we twist the propagators $z_1$ and $z_3$, but the index $n_1$ must be analytically continued to a negative value. The standard procedure is to replace the infinite sum by a Sommerfeld-Watson contour integral, and then push back the contour exposing the poles on the negative real axis. To generate the necessary pole in $n_1$ we first evaluate (5.8) at the critical point. About $(x_1, x_3)$, the amplitude (5.8) is approximately

$$A_6 \sim x_1^{-1-\alpha(t_1)} x_3^{-1-\alpha(t_3)} (1-x_1)^{-1+\alpha(s_{23})} (1-x_3)^{-1+\alpha(s_{45})} I_6,$$

(5.11)

where, after shifting $z_1$ and $z_3$, the integral becomes

$$I_6 = \prod_{r=2}^{\infty} \sum_{r=0}^{\infty} \frac{1}{r^{n_r} n_r!} \left[ \hat{s} \left( x_1^r - x_1 \right) \left( x_3^r - x_3 \right) \right]^{n_r}
\times \sum_{n_1=0}^{\infty} \frac{1}{n_1!} \frac{1}{\sum_{j=1}^{\infty} j n_j - \alpha(t)} \int_{-\epsilon}^{\epsilon} dz_1 dz_3 (\hat{s} z_1 z_3)^{n_1}.
\tag{5.12}$$

The double integral is easily performed, giving

$$I_6 = 2\hat{s}^{-1} \prod_{r=2}^{\infty} \sum_{n_r=0}^{\infty} \frac{1}{r^{n_r} n_r!} \left[ \hat{s} \left( x_1^r - x_1 \right) \left( x_3^r - x_3 \right) \right]^{n_r}
\times \sum_{n_1=0}^{\infty} \frac{y_0^{n_1+1}}{n_1!} \frac{1}{\sum_{j=1}^{\infty} j n_j - \alpha(t)} \frac{1 - (-1)^{n_1+1}}{(n_1 + 1)^2},
\tag{5.13}$$

where $y_0 = e^{2\hat{s}}$. As we will see shortly, it is crucial for the sister that $n_1$ appear as a double pole.

As a prelude to replacing the discrete variable $n_1$ by a continuous one, we must replace $n_1!$ by its gamma function representation $\Gamma(n_1 + 1)$. To analytically continue to the pole at $n_1 = -1$ we must be careful since Eq. (5.13)
vanishes for odd \( n_1 \) because of the numerator in the last factor. Using the Sommerfeld-Watson transformation to convert the sum over \( n \) into a contour integral, we get

\[
I_6 = \frac{1}{\pi i} \sum_{r=2}^{\infty} \frac{1}{r^{n_r} n_r!} \left[ \delta(x_1 - x_1) \left( x_3^r - x_3 \right) \right]^{n_r}
\]

\[
\times \int C \frac{dx}{\sin \pi x} \frac{y_0^{x+1}}{\Gamma(x+1)} \frac{1}{x + \sum_{j=2}^{\infty} jn_j - \alpha(t)} \frac{1 - (-1)^{x+1}}{(x+1)^2} + \ldots.
\]

where we have separated out, and displayed, the odd \( n \) contributions. To continue back to the pole, we must first signaturize the last factor by setting \(-1 = e^{i\pi}\). Pushing back the contour then exposes the double pole with residue

\[
I_6 = 2 \delta^{-1} \prod_{r=2}^{\infty} \frac{1}{r^{n_r} n_r!} \left[ \delta(x_1 - x_1) \left( x_3^r - x_3 \right) \right]^{n_r}
\]

\[
\times \frac{d}{dx} \left[ \frac{y_0^{x+1}}{\sin \pi x} \frac{1 - e^{i\pi(x+1)}}{\Gamma(x+1)} \frac{1}{x + \sum_{j=2}^{\infty} jn_j - \alpha(t)} \right]_{x=-1}.
\]

Since this expression vanishes when we set \( x = -1 \) in the last factor, we only need to differentiate this term. The result is \(-i\pi\). The reason why we require a double critical point should now be clear. If only \( z_1 \) or \( z_3 \) were critical, a single \( n_1 \) pole would result whose residue vanishes.

Next, in the limit \( x \to -1 \).

\[
\sin \pi x \Gamma(x + 1) \to 1.
\]

Thus,

\[
I_6 = -2i \pi \delta^{-1} \prod_{r=2}^{\infty} \frac{1}{r^{n_r} n_r!} \left[ \delta(x_1 - x_1) \left( x_3^r - x_3 \right) \right]^{n_r}
\]

\[
\times \frac{1}{-1 + \sum_{j=2}^{\infty} jn_j - \alpha(t)}.
\]

In the special case \( n_r = 0 \) for \( r = 2, 3, \ldots \), we have the nonsense pole at \( \alpha(t) = -1 \), i.e.,

\[
I_6 = -2i \pi \delta^{-1} \frac{1}{-1 - \alpha(t)}.
\]
Since the analytical continuation was along the curve described by \( \alpha_{n_1}^{n_1} | 0 \), it is clear that this pole is generated by the leading Regge trajectory \( \alpha(t) \). The analysis above shows that this pole is given by \( "(\alpha_{-1})^{-1} | 0" \). The inverse oscillator indicates the unphysical nature of this state.

Now, we must explicitly show that the pole (5.18) is canceled by a corresponding pole on the leading first sister trajectory, \( \beta(t) \). To analytically continue to this pole along \( \beta(t) \), we must convert the \( n_2 \) sum in (5.13) to a contour integral

\[
I_6 = \frac{1}{\pi i} \frac{s^{-1}}{2} \sum_{n_1=0}^{\infty} \frac{y_0^{n_1+1} 1 - (-1)^{n_1+1}}{n_1! (n_1 + 1)^2} \frac{1}{\sin(\frac{1}{2} \alpha(t) - \frac{1}{2} n_1)} \int_C \frac{dx}{\sin \pi x} \frac{1}{2 x \Gamma(x + 1)} \left[ \hat{s}(x_1^2 - x_1) (x_3^2 - x_3) \right]^x \frac{1}{n_1 + 2x - \alpha(t)},
\]

where we have set \( n_r = 0 \) for \( r = 3, 4, \ldots \). Picking up the pole at \( x = \frac{1}{2} \alpha(t) - \frac{1}{2} n_1 \) gives

\[
I_6 = 2 \frac{s^{-1}}{2} \sum_{n_1=0}^{\infty} \frac{y_0^{n_1+1} 1 - (-1)^{n_1+1}}{n_1! (n_1 + 1)^2} \frac{1}{\sin(\frac{1}{2} \alpha(t) - \frac{1}{2} n_1)} \int_C \frac{dx}{\sin \pi x} \frac{1}{2 x \Gamma(x + 1)} \left[ \hat{s}(x_1^2 - x_1) (x_3^2 - x_3) \right]^x \frac{1}{\sin(\frac{1}{2} \alpha(t) - \frac{1}{2} n_1)}. \]

Repeating the steps for \( n_1 \) and evaluating at the pole \( n_1 = -1 \), we finally obtain

\[
I_6 = -i 2 \frac{s^{\beta(t)}}{\sin \pi (\beta(t) + 1)} \frac{1}{\Gamma(\beta(t) + 2)} \left[ (x_1^2 - x_1) (x_3^2 - x_3) \right]^{\beta(t)+1}. \]

That the first pole at \( \beta(t) = -1 \) cancels the amplitude (5.18) can be seen by writing \( \Gamma(\beta(t) + 2) = (\beta(t)+1)\Gamma(\beta(t) + 1) \), and then canceling the pole coming from the sine function against the zero in \( \Gamma^{-1}(\beta(t) + 1) \). The remaining poles are represented by the states \( (\alpha_{-1})^{-1} \alpha_{-2}^{n_r} | 0 \) and cancel unphysical poles generated by the daughters of the \( \alpha(t) \) trajectory.
Repeating the six-point computation for a general critical point defined by 
\[ z_1' = x_1 \text{ and } z_3' = x_3 \] produces the \((r - 1)\text{th}\) daughter of the \(\beta(t)\) sister. The complete set of states corresponding to \(\beta(t)\) and all its daughters is given by 
\[ (x_{-1})^{-m} x_{-2}^{n_2} \mid 0 \], for \(m > 0\) and \(n_2 \geq 0\). The set can only exist in totality, and results from a complete saturation of the propagator with the oscillator \(x_{-1}\). In the present context, we see that the \(\beta(t)\) sister does not appear in either the four- or five-point function, or on the \(z_1\) and \(z_3\) propagators of the six-point function, because coupling the propagator to two on-shell states at any vertex prevents total saturation.

This analysis suggests that to obtain the second sister trajectory, \(\gamma(t)\), we must first saturate the appropriate propagator with the oscillator \(x_{-1}\), permitting \(\beta(t)\) to exist, and then with \(x_{-2}\). To verify this, we again consider the eight-point diagram of of Fig. 3. Inserting the identity operator on adjacent sides of the propagator \(z_3\), and using the four-dimensionality constraints, yields the form 
\[ A_8 \simeq \int_0^1 dz_1 dz_2 dz_4 dz_5 f(z_1, z_2, z_4, z_5) \]
\[ \times \prod_{r=1}^{\infty} \sum_{n_r=0}^{\infty} \frac{1}{r^{n_r} n_r!} \sum_{j=1}^{\infty} \frac{1}{j n_j - \alpha(t)} (\hat{s} H_r)^{n_r}, \] (5.22)
where \(t \equiv t_3\), and
\[ H_r = (z_1' - x_1)(z_2' - x_2)(z_4' - x_4)(z_5' - x_5). \] (5.23)

and where the \(x'\)s are as before.

Recall, we can obtain the leading \(\gamma(t)\) trajectory if we twist all the noncentral propagators and expand about the point \(z_1 = x_1, z_2 = x_2, z_4 = \sqrt{x_4}\), and \(z_5 = \sqrt{x_5}\). For \((x_1, x_2)\) and \((\sqrt{x_4}, \sqrt{x_5})\) to represent double critical points will also require we continue both \(n_1\) and \(n_2\) to negative values. Assuming this to
be a case, we have
\[ A_8 \sim f(x_1, x_2, \sqrt{x_4}, \sqrt{x_5}) \prod_{r=3}^{\infty} \sum_{n_r=0}^{\infty} \frac{1}{r^{n_r} n_r!} (\delta^r H_r)^{n_r} \int_{-\infty}^{\infty} dz_1 dz_2 dz_4 dz_5 \]
\[ \times \sum_{n_1=0}^{\infty} \frac{1}{n_1!} [\delta z_1 z_2 (\sqrt{x_4} - x_4) (\sqrt{x_5} - x_5)]^{n_1} \]
\[ \times \sum_{n_2=0}^{\infty} \frac{1}{2 n_2 n_2!} [4 \delta z_4 z_5 \sqrt{x_4 x_5} (x_1^2 - x_1)(x_2^2 - x_2)]^{n_2} \frac{1}{\sum_{j=1}^{\infty} j n_j - \alpha(t)}. \]
(5.24)

Each pair of integrals is the same form as in (5.12). The first pair generates a double pole at \( n_1 = -1 \) and moves us onto a \( \beta(t) \) trajectory. Subsequently, the second pair gives a double pole at \( n_2 = -1 \) which now transfers us to a \( \gamma(t) \) trajectory.

Integrating and analytically continuing to \( n_1 = -1 \) we immediately find
\[ A_8 \sim -4i\pi \left(2 \delta^2 x_4 x_5 (x_1^2 - x_1)(x_2^2 - x_2)(1 - \sqrt{x_4})(1 - \sqrt{x_5}) \right)^{-1} \]
\[ \times f(x_1, x_2, \sqrt{x_4}, \sqrt{x_5}) \prod_{r=3}^{\infty} \sum_{n_r=0}^{\infty} \frac{1}{r^{n_r} n_r!} (\delta^r H_r)^{n_r} \]
\[ \times \sum_{n_2=0}^{\infty} \frac{v_0^{n_2+1}}{n_2!} \frac{1 - (-1)^{n_2+1}}{(n_2 + 1)^2} \frac{1}{-1 + \sum_{j=2}^{\infty} j n_j - \alpha(t)}, \]
where the exact form of \( v_0 \) is not important. We can approach the final state \((\alpha_{-1} \alpha_{-2})^{-1} | 0 \rangle\) by moving along either the leading \( \beta(t) \) or \( \gamma(t) \) trajectories. The nonsense poles obtained in two cases must cancel. For the first path, we set \( n_r = 0 \) for \( r = 2, 3, \ldots, \) and perform a Sommerfeld-Watson transformation on \( n_2 \), to get
\[ A_8 \sim -4\pi^2 \left(2 \delta^2 x_4 x_5 (x_1^2 - x_1)(x_2^2 - x_2)(1 - \sqrt{x_4})(1 - \sqrt{x_5}) \right)^{-1} \]
\[ \times f(x_1, x_2, \sqrt{x_4}, \sqrt{x_5}) \frac{1}{-3 - \alpha(t)}. \]
(5.25)

Similarly, for \( n_3 \neq 0 \), sliding down the \( \gamma(t) \) trajectory we obtain the result
\[ A_8 \sim -\frac{4}{3}\pi^2 \left(2 x_4 x_5 (x_1^2 - x_1)(x_2^2 - x_2)(1 - \sqrt{x_4})(1 - \sqrt{x_5}) \right)^{-1} \]
\[ \times f(x_1, x_2, \sqrt{x_4}, \sqrt{x_5}) \delta^{\gamma(t)} H_3^{\gamma(t)+2} \frac{1}{\sin \pi(\gamma(t) + 2)} \frac{1}{\Gamma(\gamma(t) + 3)}. \]
(5.27)
where, for $\gamma(t) = -2$, this cancels the pole (5.26).

Extending these results to the most general case, suggests that the $r$th sister trajectory forms when the propagator becomes successively saturated by the oscillators $a_m$, starting with $m = 1$, and eventually reaching $m = r-1$. In another words, to get to the trajectory $a_m(t)$ we begin by moving down either a leading or daughter $a(t)$ trajectory curve to either a leading or daughter $\beta(t)$ trajectory, which we reach by analytical continuation. etc. The resultant sister and its daughter trajectories are represented by open string states of the form

$$(\alpha_1)^{-m_1} \cdots (\alpha_{-r+1})^{-m_{r-1}}(\alpha_{-r})^{m_r} | 0 \rangle, \quad \text{for } m_1, \ldots, m_{r-1} > 0, n_r \geq 0.$$  (5.28)

where the leading trajectory is given by $m_1 = \ldots = m_{r-1} = 1, n_r = 0$. By analogy we can immediately write down the corresponding closed string sister states by replacing the open string oscillator $a_{-r}$ with the closed string oscillators $a_{-r} \alpha_{-r}$ everywhere.

An important point that needs to be stressed here for applying the procedure we have presented, is that it be possible to completely isolate the sister propagator. In the case of the double-loop four-point interaction, this criterion adds justification to our approach in Chap. 3 where we factorized the amplitude so that we could treat as individual objects the two loops and the connecting propagator. In the state analysis approach, projecting the physical states onto the connecting propagator gives

$$A_{(4;2)} = \prod_{r=1}^{\infty} \sum_{n_r=0}^{\infty} \frac{1}{r^{n_r} n_r!} \sum_{j=1}^{\infty} \frac{1}{j n_j - a(t)} \int d\mu \int_0^1 \prod_{i=1}^{4} dz_i,$$

$$\times (0 | \exp \left[ -\frac{1}{2 \ln k_1} + C_L \right] (\alpha_{-1})^{n_1} \cdots (\alpha_{-r})^{n_r} | 0 \rangle$$

$$\times (0 | (\alpha_{1})^{n_1} \cdots (\alpha_{r})^{n_r} \exp \left[ -\frac{1}{2 \ln k_2} + C_R \right] | 0 \rangle.$$  (5.29)
After simplifying, the calculation leading to the sister proceeds exactly as in the six-point case given earlier in this chapter, and reproduces the results of Chap. 3.

Exposing the open string sister in the four-point closed string diagram of Fig. 8 presents a new difficulty, however, since we must look for a triple Regge pole that also is not a leading order term. The expression

\[
A_{G^2} = \left(\frac{1}{4\pi}\right)^2 \int_{|z_i| \leq 1} d^2 z_1 d^2 z_2 \langle 0, p_4 | V^\dagger(p_3, z_1^{-1}, \bar{z}_1^{-1}) \times o(0 | Y(A^\dagger, a) | 0)_{\text{cl}}
\]

\[
\times I\Delta_o I_{\text{cl}}(0 | Y(A, a^\dagger) | 0)_{\text{o}} \times V(p_2, z_2, \bar{z}_2) | 0, p_1)_{\text{cl}}.
\]

(5.30)

reduces, after some algebra, to

\[
A_{G^2} = \left(\frac{1}{4\pi}\right)^2 \prod_{r=1}^{\infty} \sum_{n_r=0}^{\infty} \frac{1}{r^{n_r} n_r!} \sum_{j=1}^{\infty} \frac{1}{j n_j + 3 - 2\alpha(t)} \int_{|z_i| \leq 1} d^2 z_1 d^2 z_2
\]

\[
\times |z_1 z_2|^{-2-\alpha(t)} \left[-2p_2 \cdot p_3(z_1^{r/2} + \bar{z}_1^{r/2})(z_2^{r/2} + \bar{z}_2^{r/2})^\text{even} \right]^{n_r}
\]

\[
\times \left[2p_2 \cdot p_3 r^2 \sum_{k,m=0}^{\infty} C_{km}^{(1)} C_{nm}^{(1)}(z_1^k - \bar{z}_1^k)(z_2^n - \bar{z}_2^n) \delta_{2m+1}^r \right]^{n_r}.
\]

(5.31)

We used the fact here that the even and odd oscillator parts can be treated separately. To obtain the open string \(\beta_o(t)\) trajectory we set \(n_r = 0\) for \(r = 3, 4, \ldots\), which allows the sums to carried out. In the high energy limit, we find

\[
A_{G^2} = \left(\frac{1}{4\pi}\right)^2 \prod_{n_1=0}^{\infty} n_1! \sum_{n_2=0}^{\infty} \frac{1}{2^{n_2} n_2! n_1 + 2n_2 + 3 - 2\alpha(t)} \int_{|z_i| \leq 1} d^2 z_1 d^2 z_2
\]

\[
\times |z_1 z_2|^{-2-\alpha(t)} \left[-s(z_1 + \bar{z}_1)(z_2 + \bar{z}_2) \right]^{n_2}
\]

\[
\times \left[s \left((1 - z_1)^{1/2} - (1 - \bar{z}_1)^{1/2}\right) \left((1 - z_2)^{1/2} - (1 - \bar{z}_2)^{1/2}\right) \right]^{n_1}.
\]

(5.32)

Of course, using the integral representation for the propagator we can easily recover the corresponding expression of Chap. 4.
Now set \( z = \rho e^{i\theta} \). Expanding about the critical points at \( \theta = 0 \) and \( \pi \) gives

\[
A_{G^2} = 2 \sum_{n_1=0}^{\infty} \frac{1}{n_1!} \sum_{n_2=0}^{\infty} \frac{2^{n_2}}{n_2!} \frac{1}{n_1 + 2n_2 + 3 - \alpha(t)} \int_0^1 d\rho_1 d\rho_2 (\rho_1 \rho_2)^{n_1+n_2-1-\alpha},
\]

\[
\times s^{n_1+n_2} \int_{-\epsilon}^{\epsilon} \frac{d\theta_1}{4\pi} \frac{d\theta_2}{4\pi} (\theta_1 \theta_2)^{n_1} \left( \left(1 - \rho_1\right)^{-\frac{1}{2}} \left(1 - \rho_2\right)^{-\frac{1}{2}} \right)^{n_1}
+ \left(1 + \rho_1\right)^{-\frac{1}{2}} \left(1 + \rho_2\right)^{-\frac{1}{2}} \right)^{n_1} (\rho_1 n_1 + 2) \right),
\]

(5.33)

where the first term in the last factor is for \( \theta_1 \) and \( \theta_2 \) being expanded about the same value, while the second term is the converse case. The \( \theta \) integrals are executed as before, giving \( (1 + (-)^{n_1})^2 \epsilon^{2n_1+2}/(n_1 + 1)^2 \). Each \( \rho \) integration produces a factor \( B(n_1 + n_2 - \alpha t, 1 - \frac{1}{2}n_1) \). Combining the terms then produces the factor \((1 + (-)^{n_1+n_2})\). This leads to the result

\[
A^{k=2} = \frac{1}{4\pi^2} \sum_{n_1=0}^{\infty} \frac{1}{n_1!} \sum_{n_2=0}^{\infty} \frac{2^{n_2}}{n_2!} \frac{1}{n_1 + 2n_2 + 3 - \alpha(t)} \epsilon^{2n_1+2}
\]

\[
\times B(n_1 + n_2 - \alpha(t), 1 - \frac{1}{2}n_1)^2 (1 + (-)^{n_2}) (1 + (-)^{n_1}) \frac{\epsilon^{2n_1+2}}{(n_1 + 1)^2}.
\]

(5.34)

where we have used \((1 + (-)^{n_1})^2 = 2(1 + (-)^{n_1})\).

Utilizing the Sommerfeld-Watson transformation, the residue due to the double pole at \( n_1 = -1 \) yields the result

\[
A^{k=2} = (-i\pi) \frac{1}{4\pi^2} \sum_{n_2=0}^{\infty} \frac{2^{n_2}}{n_2!} \frac{1 + (-)^{n_2}}{2 + 2n_2 - 2\alpha(t)} B(-1 + n_2 - \alpha(t), 3/2)^2 s^{1+n_2}.
\]

(5.35)

Writing the Beta function in terms of \( \Gamma \) functions, and pulling out the first three poles from one of the \( \Gamma \)'s gives

\[
A^{k=2} = -\frac{i}{8\pi} \sum_{n_2=0}^{\infty} \frac{2^{n_2}}{n_2!} \frac{1 + (-)^{n_2}}{(1 + n_2 - \alpha(t))^3} s^{1+n_2}
\]

\[
\times \left[ \frac{\Gamma(2 + n_2 - \alpha(t))\Gamma(3/2)}{(-1 + n_2 - \alpha(t))(n_2 - \alpha(t))\Gamma(\frac{1}{2} + n_2 - \alpha(t))} \right]^2.
\]

(5.36)
Thus, we have recovered the triple pole at $n_2 = \alpha(t) - 1$. Computing the residue by taking the second derivative of the energy factor, gives the final result

$$A^{k=2} = -\frac{i9}{\pi}2^{-9+\beta_o(t)} \frac{1 - e^{-i\pi\beta_o(t)}}{\sin\pi(\beta_o(t) + 1)} \frac{1}{\Gamma(\beta_o(t) + 2)} s^{\beta_o(t) - 1} \ln^2 s. \quad (5.37)$$

To show that these poles cancel, we again start with (5.34) but now pick up the single pole at $n_1 = 2\alpha(t) - 3 - 2n_2$ to get

$$A^{k=2} = \frac{1}{4\pi^2} \sum_{n_2=0}^{\infty} \frac{2^{n_2}}{n_2!} \left[ \frac{\Gamma(3 - n_2 + \alpha(t))\Gamma(5/2 - \alpha(t) + n_2)}{\Gamma(-1/2)} \right]^2 \times s^{2\alpha(t) - 3 - n_2} (1 + (-)^{n_2})(1 + (-)^{2\alpha(t) - 3 - 2n_2}) \frac{\epsilon^{4\alpha(t) - 4 - 4n_2}}{(2\alpha(t) - 2 - 2n_2)^2}. \quad (5.38)$$

We must take the residue of the quadruple pole at $n_2 = \alpha(t) - 1$, which will give a factor of $\frac{1}{3!}$. For this, we take two derivatives of the energy factor and, to get a nonvanishing result, one derivative of the factor $(1 + (-)^{2\alpha(t) - 3 - 2n_2})$, which can be done 3 ways. This leads to

$$A^{k=2} = \frac{1}{3!} \frac{1}{8\pi^2} \frac{1}{\sin\pi(\alpha(t) - 1)} \frac{1}{\Gamma(\alpha(t))} \left[ \frac{1}{\frac{1}{2}\Gamma(3/2)} \right]^2 (2s)^{\alpha(t) - 2} (2i\pi \ln^2 s), \quad (5.39)$$

which reduces to the negative of (5.37).
In this thesis we have indicated that the Cerulus-Martin bound may not be violated in string theory if one includes higher order corrections to the tree diagram. To complete the proof requires that the entire perturbation series be summed to determine if the coefficients of the amplitudes have any effect upon the result. It is not clear, however, how to take the fixed angle limit in the high energy analysis employed above. Instead, the proper approach may be to adopt the techniques used by Gross and Mende.\textsuperscript{4} Basically, for closed strings, this means searching for saddle points on an \( N \) sheeted Riemann surface defined by an appropriate algebraic curve. To have the sisters produce the dominant behavior would require that we consider the limiting situation where the Riemann surface is divided in two, separated by a thread representing the sister propagator. The hope is that the uncontrollable higher order corrections which plagued the work of Ref. 4 would now be absent.

Although this may remove one of the objections to locality, it should not imply that in any way have we shown that string theory is, or can be, a local theory. There still remains serious objections which may be more difficult, if not impossible, to overcome. For example, in string field theory, Eliezer and Woodard\textsuperscript{6} note that the cubic formulation of the field theory produces an infinite number of Abelian solutions.\textsuperscript{32} This causes a breakdown of the initial value problem since it requires an infinite amount of initial data. They show that attempts to restore this loss of predictability result in acausal behavior,
which, again, leads to nonlocality. Another problem in string field theory, is that the individual elements of the perturbative S-matrix still violate the CM bound. Possibly, with the help of the sisters, one can find a local field theory where strings are produced non-pertubartively and appear as bound states. This would probably resolve most of the locality problems in the field theory. We should point out, though, that it is not even clear if field theory should be the fundamental formulation for strings.

Finally, let us comment on a physical interpretation for the sister states. Recall, the Regge slope $\alpha'$ is related to the string tension $T$, or energy per unit length, by

$$\alpha' = \frac{1}{\pi T}. \quad (6.1)$$

This shows that, for instance, the first sister $\beta(t)$ has twice the tension of $\alpha(t)$. We visualize this occurring by bending over the string once to create a double strand, giving a 'folded' string. This picture is in accord with a reduction of the fundamental length scale $l = \sqrt{\alpha'}$. The notion of folded strings originally dates back to the early 1970's where it was noted that pure states of the form $\alpha'^n | 0\rangle$ have Regge slope reduced by a factor of $\frac{1}{r}$. Thus, the state analysis of Chap. 5 furthers the identification of the sister trajectories with folded strings.

In conclusion, the motivation for studying sister trajectories is that they may eventually lead to a useful description of the short distance behavior of string theory. Presently, there is a growing belief that the current version of string theory is nonlocal at a fundamental level. The sister trajectories, and their interpretation as folded strings, may be the necessary ingredient for reformulating string theory to produce a local theory.
APPENDIX
WEIGHT DIAGRAMS AND LAX OPERATORS

Recently, matrix models have received a great deal of attention as nonperturbative descriptions of string theory.\textsuperscript{7,8,9} Since initial advancements, progress has proceeded in many different directions. In particular, Douglas\textsuperscript{33} has shown that the limited number of known matrix model solutions can be derived from the Lax pair formalism usually associated with the KdV equations. This identification with integrable systems greatly increases the number of classifiable matrix models since it was shown a long time ago\textsuperscript{34} that Lax operators are associated with affine Lie algebras. For example, the models discussed by Douglas are related to the canonical representations of $A_n^{(1)}$. More recently, Di Francesco and Kutasov\textsuperscript{35,36} have discussed $D_n^{(1)}$ based matrix models which the standard matrix techniques\textsuperscript{37,38,39} have yet to solve. Thus, it may be worthwhile to focus on the integrable systems approach.

Several approaches to constructing the Lax operators have been developed. The matrix procedure discussed by Drinfel’d and Sokolov\textsuperscript{34} defines first a matrix eigenvalue equation. The system incorporates knowledge of the Cartan subalgebra and root system of some embedding affine Lie algebra $\hat{G}$. Starting with an affine Lie algebra facilitates the construction of an integrable system from the resulting Lax pair operators. To fix the gauge invariance in the matrix system, the gradation conventions of Drinfel’d and Sokolov require that one of the simple roots, say the $m^{th}$, must be removed from the affine system. The resulting system is denoted by $(\hat{G}, c_m)$. This is equivalent to deleting the $m^{th}$ Dynkin vertex. For the most part, Drinfel’d and Sokolov choose the “canoni-
cal” gauge in which to express the coordinate dependent terms. In this gauge, Lax operators generate the regular KdV hierarchy equations.

The modified KdV(mKdV) equations can be generated by expressing the coordinate term $q(x)$ in the “diagonal” gauge. The canonical Lax operators can then be recovered using the well-known Miura transformations. The diagonal gauge is technically simpler than the canonical gauge. Furthermore, the final Lax operator is in a factorized form which has been used to quantize the theory.\(^{40}\)

In this appendix our focus will be on the explicit construction of the Lax (pseudo)differential operators in the diagonal gauge using a simple diagrammatic technique.\(^ {41}\) In most cases this technique arrives at these operators much quicker than a direct application of the scheme of Drinfel’d and Sokolov. Furthermore, the scheme also applies to higher representations of the embedding affine Lie algebra. In the first section we briefly review the construction of weight diagrams corresponding to representations of affine and non-affine Lie algebras. From there we review the matrix method of Drinfel’d and Sokolov for building Lax operators. Next is a presentation of our method, which replaces the matrix procedure with a scheme utilizing cyclic weight diagrams of representations of affine Lie algebras. We then present a proof that the diagrammatic algorithm produces the correct Lax operator. Finally, we discusses the generalization to Lax operators based on supersymmetric affine Lie algebras.

**Review of Weight Diagrams**

As noted in the introduction, each Lax operator can be associated with a representation of some affine Lie algebra. Thus, in this section, we give a
brief review for constructing weight diagrams corresponding to these representations.

Recall, one can associate uniquely to every irreducible representation of a basic Lie algebra a highest weight vector.\textsuperscript{42} For each highest weight one can construct a weight diagram which encodes all relevant information concerning the particular representation studied, \textit{e.g.}, from it one can build explicit matrix representations of the generators of the Cartan subalgebra as well as the various raising and lowering operators. The level of a weight is the number of lowering operators applied to the highest weight which produces that weight. Finally, the height $\gamma$ of the weight diagram is the level of the lowest weight.

Weight diagrams are generated by subtracting rows of the Cartan matrix initially from the highest weight vector written in the Dynkin basis. Rules of construction can be summed up as follows:

1. Subtract the $i^{th}$ row of the Cartan matrix $n$ times from a weight vector whose $i^{th}$ component has a positive value $n$.

2. When weight vectors have more than one positive component, subtract all possible permutations of the appropriate Cartan rows.

A theorem due to Dynkin\textsuperscript{43} states that the final weight diagram is always "spindle shaped". In other words: i) the number of weight vectors at the level $k$ is equal to the number at level $\gamma - k$; ii) the number of weights at level $k + 1$ is greater than or equal to the number at level $k$ for $k < \frac{\gamma}{2}$.

For an explicit example consider the algebra $A_2$. Though this is almost a trivial case, the results will be useful for the next section. The Dynkin diagram is given by

\[
\begin{array}{c}
\circ \\
1 \quad 0
\end{array}
\]
where, recall, the single bar represents 120°. The Cartan matrix is then easily found to be

\[
A = \begin{pmatrix}
2 & -1 \\
-1 & 2
\end{pmatrix}.
\]  \hspace{1cm} (A.1)

The highest weight vector of the fundamental representation is (10). Since a positive one appears in the first place, we subtract the first row of the Cartan matrix one time. This gives the weight (-11). Now, due to the one in the second place, we subtract the second row of the Cartan matrix once to get (0 -1). This completes the process since no positive components remain. The result is the height two weight diagram

\[
\begin{array}{c}
(10)_1 \\
\downarrow 1 \\
(-11)_2 \\
\downarrow 1 \\
(0 -1)_3
\end{array}
\]  \hspace{1cm} (A.2)

where the subscripts on the weight vectors indicate a counting of the vectors. The ones adjacent to the arrows represent the normalization factors of the corresponding negative simple roots. These values are fixed by the commutation relations of the Lie algebra. To simplify our diagrams, we will not display values of unity. Later, we will see that the procedure for building weight diagrams is slightly modified in the supersymmetric case.

For an affine Lie algebra, since there exists a linear combination among simple roots, weight diagrams of affine representations generally have infinite extent. However, some affine representations give cyclic weight diagrams of finite extent. In fact, it is these cyclic cases that are crucial to the scheme below. To produce a cyclic weight diagram, the affine component which is appended to the highest weight vector of the underlying non-affine Lie algebra, unlike the non-affine weight components, may have to be assigned a negative value.
Figure 10. The \((-110)\) representation of \(A_1^{(1)}\). (a) Dynkin diagram; (b) Cartan matrix; (c) Cyclic weight diagram. The dashed arrow is the deleted root.

As an explicit example, consider the non-twisted affine algebra \(A_2^{(1)}\). To generate the cyclic weight diagram corresponding to the canonical representation, start with the weight vector \((-110)\), where \(-1\) corresponds to the affine root. Figure 10 gives the Dynkin diagram and subsequent Cartan matrix which then generates the displayed resultant weight diagram.

This particular cyclic weight diagram can further be thought of as the affine extension of the highest weight diagram based on the fundamental representation of the basic Lie algebra \(A_2\). This is easy to see by removing everywhere the component due to the affine root. However, this is not always the case. For example, Fig. 11 displays the cyclic weight diagram constructed with the weight vector \((-211)\), where now the affine component is \(-2\). Although (11)
Figure 11. Cyclic weight diagram of $A_1^{(1)}$ from the weight $(-211)$. The dashed arrows are the deleted root.

\[
\begin{array}{c}
(1 1) \\
(1 -2) \quad (-2 1) \\
(2 -1 -1)
\end{array}
\]

Figure 12. Highest weight diagram of the adjoint representation of $A_2$, (11).

generates the highest weight diagram of the adjoint representation of $A_2$, we see by comparing with Fig. 12 that the affine extension contains an extra zero weight $(000)$. 
In general, the affine component in the affinely extended vector, associated with the highest weight vector of a basic Lie algebra, will always be negative. However, we will give an example below showing that some supersymmetric cases require positive affine components.

**Standard Construction of Lax Operators**

The Lax operator $L(x,t)$ is defined to be linear and Hermitian. Furthermore, it satisfies the characteristic equation

$$L(x,t)\phi(x,t) = \mu \phi(x,t), \quad (A.3)$$

where the eigenvalue $\mu$ is required to be constant under nonlinear evolution. In other words, the nonlinear behavior of the eigenfunctions $\phi(x,t)$ are governed by some operator $A(t)$, which may be nonlinear, via the equation

$$\frac{\partial \phi(x,t)}{\partial t} = A(t)\phi(x,t). \quad (A.4)$$

Furthermore, $A(t)$ enters into the differential scalar Lax equation

$$\frac{\partial L(x,t)}{\partial t} = [A(t), L(x,t)], \quad (A.5)$$

which generates the integrable KdV equations.

The matrix construction of Lax operators utilizing generators of some embedding affine Lie algebra $\hat{G}^{(k)}$, reviewed by Drinfel’d and Sokolov,\textsuperscript{34} begins with a matrix operator of the form

$$\mathcal{L} = I \frac{\partial}{\partial x} + \Lambda + q(x). \quad (A.6)$$

where $I$ denotes the $N \times N$ dimensional unit matrix, and to simplify notation we have suppressed the argument $t$. The third term is discussed below. The second term is generated by the negative simple roots $E_i$ of the embedding
affine Lie algebra. In the gradation conventions of Drinfel’d and Sokolov, we have the circulant matrix
\[ \Lambda = \sum_{i=0}^{r} c_i E_i \]  \hspace{1cm} (A.7)

The procedure is then to reduce the system of linear equations given by the kernel matrix equation
\[ \mathcal{L} \vec{\psi}(x) = 0, \]  \hspace{1cm} (A.8)
where \( \vec{\psi} = (\psi_1, \ldots, \psi_N) \), to the linear differential eigenvalue equation (A.3), where the vacuum solution \( \phi \) is a function of the components of the eigenfunction \( \vec{\psi} \). Drinfel’d and Sokolov show that such reduction is possible if one removes a simple root, say the \( m^{th} \), from the affine root system. They denote this situation \( (\mathcal{G}^{(k)}, c_m) \), which is in the homogeneous or standard gradation. The coefficients in (A.7) are then assigned the values \( c_{i \neq m} = 1 \) and \( c_m = \lambda \), where \( \lambda \) is a constant function of the spectral parameter \( \mu \).

Removing an element from the simple root system is equivalent to deleting the corresponding vertex from the Dynkin diagram. Thus, when an extremal vertex is deleted, the system \( (\mathcal{G}^{(k)}, c_m) \) represents a single residual basic Lie algebra. Removing the affine vertex obviously gives \( \mathcal{G} \). On the other hand, deleting internal vertices splits the Dynkin diagram into two sections, corresponding to a pair of basic Lie algebras. For example, splitting \( A_{2n}^{(2)} \) at the \( m^{th} \) vertex gives Lax operators in the \( B_{n-m} \) and \( C_m \) series. Furthermore, the (pseudo) differential operator associated with the \( D_n \) series is derived using the embedding algebra \( D_n^{(1)} \). In both situations, a pair of (pseudo)differential operators is found whose product gives the Lax operator \( L \) of Eq.(A.5).

To determine the exact form of the vacuum solution \( \phi \) recall that negative simple roots are lowering operators on system eigenstates. Further, removal of a root in the affine system produces the simple root system of a non-affine
Lie algebra. Thus, due to the linear combination among the roots of the affine system, one root must be singled out to act as a conventional state raising operator. This role is given to the removed root. Thus, the vacuum eigenstate will be annihilated by a vacuum projection operator $\Delta^-$ defined by

$$\Lambda = \sum_{i=0}^r E_i \mp m + \lambda E_m \equiv \Delta^- + \lambda E_m . \quad (A.9)$$

This requirement fixes the scalar vacuum solution $\phi$ by setting it equal to a linear combination of the components of $\tilde{\psi}$ such that

$$\Delta^- \tilde{\psi} = 0 . \quad (A.10)$$

is satisfied. A direct relation between the scalar operator $L$ and the matrix operator $\mathcal{L}$ will be given in the next section.

For the kernel equation (A.8) to produce a unique solution, we require that the number of independent degrees of freedom equal the rank of the embedding affine Lie algebra $\hat{\mathfrak{g}}^{(k)}$, or equivalently the residual system $(\hat{\mathfrak{g}}^{(k)}, c_m)$. The extra degrees of freedom generate gauge invariance. To fix the gauge invariance, one must find a matrix operator $S(x)$ that enforces the gauge transformation

$$\mathcal{L}_0 = e^{ad S} \mathcal{L} , \quad (A.11)$$

where $ad$ denotes the adjoint mapping. The gauge freedom in Eq.(A.8) allows one freedom in determining the form of the coordinate dependent term $q_0(x)$, i.e.,

$$\mathcal{L}_0 = I \frac{\partial}{\partial x} + \Lambda + q_0(x) . \quad (A.12)$$

Drinfel’d and Sokolov find the sufficient condition that $S \in C^\infty(R, \eta)$, where $\eta$ is generated by the positive simple roots $F_i, i \neq m$.

Many authors, including Drinfel’d and Sokolov, work most frequently in the “canonical” gauge. However, in this paper we choose to work in their
“diagonal” gauge which has the form

\[
q^{\text{diag}}(x) = \begin{pmatrix}
q_1 & 0 & \cdots & 0 & 0 \\
0 & q_2 & \cdots & 0 & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & \cdots & q_{N-1} & 0 \\
0 & 0 & \cdots & 0 & q_N
\end{pmatrix}.
\]  

(A.13)

This gauge leads to the convenient form

\[
q^{\text{diag}} = \sum_{i=0}^{r} v_i(q_1, q_2, \ldots, q_N)H_i,
\]

which is in the canonical or principal gradation. Here, \(H_i\) are the generators of the Cartan subalgebra and the functions \(v_i\) are linear combinations of the elements \(q_i\). In this gauge, the gauge term \(q^{\text{diag}}\) associated with \((\hat{g}^{(k)}, c_m)\) is the special case where the sum excludes \(i = m\). The Lax operator \(L^{\text{diag}}\) generates the mKdV equations, and is related to \(L^{\text{can}}\) via the well-known Miura transformations.

**Diagrammatic Construction of Lax Operators**

To exploit gauge invariance of the Lax operators, one should choose a \(q(x)\) gauge most suited to one’s needs. Here, we are interested in developing a diagrammatic scheme for constructing \(L\). In this regard, the diagonal gauge proves more useful than the other choices. In this section, we will demonstrate how the diagonal gauge allows one to build Lax operators directly from cyclic weight diagrams of representations of affine Lie algebras.

To motivate the algorithm, we first review the construction of \(L\) by solving the matrix system \(L\vec{\psi} = 0\). For the present discussion, it will be sufficient to consider embedding algebras of the form \((\hat{g}^{(k)}, c_0)\) where the affine vertex is deleted. Thus, the diagonal gauge simply reduces to the form

\[
q^{\text{diag}}(x) = \sum_{i=1}^{r} v_i(x)H_i,
\]

(A.15)
where we have excluded $H_0$ from the sum.

Consider again the canonical representation of the embedding affine Lie algebra $(A_2^{(1)}, c_0)$ presented in Fig. 10. Matrix representations of the Cartan matrix can be read-off from the cyclic weight diagram. The matrix element $(H_i)_{jj}$ is extracted from the $i^\text{th}$ element of the $j^\text{th}$ weight vector, while the off-diagonal elements are set to zero. The matrix entry of the negative simple root $(E_i)_{jk}$ is assigned its normalization factor if the $k^\text{th}$ weight vector branches into the $j^\text{th}$ weight vector as a result of subtracting the $i^\text{th}$ row of the Cartan matrix in the process. The other entries are by default zero. Thus, the matrix representations of the simple roots are easily found to give

$$
\Lambda = \begin{pmatrix}
0 & 0 & \lambda \\
1 & 0 & 0 \\
0 & 1 & 0
\end{pmatrix},
$$

where the effect of the affine root, indicated in Fig. 10 by the dashed arrow line, is assigned the value $\lambda$. Plugging these values into the kernel equation (A.8) produces the system of equations

$$
[\partial + v_1]\psi_1 = -\lambda \psi_3,
$$

$$
[\partial - v_1 + v_2]\psi_2 = -\psi_1, \quad (A.17)
$$

$$
[\partial - v_2]\psi_3 = -\psi_2.
$$

Here, on the right-hand side we have placed the terms due to the matrix $\Lambda$.

The vacuum condition (A.10) determines the scalar function to be $\phi = \psi_3$. Thus, we must solve by starting with the last equation. First, we multiply this equation through by $[\partial - v_1 + v_2]$, and then eliminate $\psi_2$ using the second equation. Then, multiplying through by $[\partial + v_1]$ and using the top equation gives the scalar Lax eigenvalue equation

$$
L(A_2^{(1)}, c_0)\psi_3 = [\partial + v_1][\partial - v_1 + v_2][\partial - v_2]\psi_3
$$

$$
= -\lambda \psi_3. \quad (A.18)
$$
where the spectral parameter is given by $\mu = -\lambda$. Imposing the field redefinitions

$$q_1 = v_1, \quad q_2 = v_2 - v_1.$$  \hspace{1cm} (A.19)

gives the standard form

$$L^{(A_2^{(1)}, c_0)} = [\partial + q_1][\partial + q_2][\partial - q_1 - q_2].$$  \hspace{1cm} (A.20)

This example exhibits a common feature relevant for our scheme below. When the vacuum condition (A.10) requires the scalar eigenfunction to be given by a single component of the eigenfunction, say $\phi = \psi_i$, then the resulting characteristic equation satisfies

$$L\psi_i = \mu \psi_i.$$  \hspace{1cm} (A.21)

Consequently, the system reduction must start with the $i^{th}$ equation in the matrix system, and proceed upward till the top equation is reached. If $i \leq N$ the process continues with the bottom equation and moves upward until the $i^{th}$ equation is reached again. We shall refer to this case as trivial since the corresponding cyclic affine weight diagram is linear, containing no branch points. A second feature brought out in this example, is that the number of factors in the resultant Lax operator (A.20) is equal to the number of weights in the weight diagram. Unfortunately, this is valid only for trivial cases. Nevertheless, this last observation is key to our scheme.

To highlight one more property of the general procedure we turn to a non-trivial example. For this, we require a representation of an affine Lie algebra whose cyclic weight diagram has at least one branching point. Thus, consider the canonical representation of the affine algebra ($D_4^{(1)}, c_0$). Fig. 13 presents the Dynkin diagram, Cartan matrix and corresponding cyclic weight diagram.
which has two branch points. Reading off from the weight diagram gives

\[
\Lambda = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & \lambda & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & \lambda & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
\end{pmatrix}
\tag{A.22}
\]

The branch points have manifested themselves by placing more than one non-zero entry in the second and sixth rows. Now, further reading off the elements...
of the Cartan matrices gives the system of equations

\[ [\partial + v_1] \psi_1 = -\lambda \psi_7, \]
\[ [\partial - v_1 + v_2] \psi_2 = -\psi_1 - \lambda \psi_8, \]
\[ [\partial - v_2 + v_3 + v_4] \psi_3 = -\psi_2, \]
\[ [\partial - v_3 + v_4] \psi_4 = -\psi_3, \]
\[ [\partial + v_3 - v_4] \psi_5 = -\psi_3, \]
\[ [\partial + v_2 - v_3 - v_4] \psi_6 = -\psi_4 - \psi_5, \]
\[ [\partial + v_1 - v_2] \psi_7 = -\psi_5, \]
\[ [\partial - v_1] \psi_8 = -\psi_7. \]  

(A.23)

The vacuum condition (A.10) produces two distinct solutions, \( \psi_8 \) and the linear combination \( \psi_4 - \psi_5 \). Here, we consider the first case. Proceeding as before, we eliminate \( \psi_6 \) and \( \psi_7 \) in the last two equations to get

\[ [\partial + v_2 - v_3 - v_4][\partial + v_1 - v_2][\partial - v_1] \psi_8 = -\psi_4 - \psi_5. \]  

(A.24)

Now, we encounter a well-known technical problem not found in the trivial case. The components \( \psi_4 \) and \( \psi_5 \) can not both be simultaneously eliminated since the expressions \( [\partial - v_3 + v_4] \) and \( [\partial + v_3 - v_4] \) do not commute. This dilemma is directly linked to the fact the corresponding cyclic weight diagram has a branch point connecting the fourth and fifth weights to a single weight located below them.

To overcome this obstacle the pseudo-differential operator \( \partial_x^{-1} \) must be introduced. Its operation on any function \( f(x) \) is given by the expansion

\[ \partial_x^{-1} f(x) = \sum_{i=0}^{\infty} (-1)^i f^{(i)}(x) \partial_x^{-1-i}. \]  

(A.25)

Utilizing the pseudo-differential operator, we rewrite the fourth equation in (A.23) as

\[ \psi_4 = -[\partial - v_3 + v_4]^{-1} \psi_3. \]  

(A.26)
Thus, the combined effect of the bottom five equations is

\[
\psi_3 = \left( [\partial - v_3 + v_4]^{-1} + [\partial + v_3 - v_4]^{-1} \right) \left[ [\partial + v_2 - v_3 - v_4]^{-1} \times [\partial + v_1 - v_2] [\partial - v_1] \psi_8. \right.
\]

(A.27)

\[
A \text{ helpful identity we use repeatedly is}
\]

\[
\{ A^{-1} + B^{-1} \}^{-1} = \{ A^{-1}[A + B] B^{-1} \}^{-1} = B[A + B]^{-1} A.
\]

(A.28)

When applied to Eq. (A.27), a cancellation occurs among the \( v_i \)'s appearing in the curly brackets. This simplifies the expression to

\[
\psi_3 = \frac{1}{2} \left[ \partial - v_3 + v_4 \right] \left[ \partial + v_3 - v_4 \right] \left[ \partial + v_2 - v_3 - v_4 \right] \left[ \partial + v_1 - v_2 \right] [\partial - v_1] \psi_8. \quad (A.29)
\]

Continuing, incorporating the next two equations in (A.23) requires a second application of the relation (A.28). Finally, the Lax operator based on \((D_4^{(1)}, c_0)\) with vacuum \( \psi_8 \) is

\[
L = \frac{1}{4} \left[ \partial + v_1 \right] \left[ \partial - v_1 + v_2 \right] \left[ \partial - v_2 + v_3 + v_4 \right] \left[ \partial - v_3 + v_4 \right] \left[ \partial + v_3 - v_4 \right] \left[ \partial + v_2 - v_3 - v_4 \right] [\partial + v_1 - v_2] [\partial - v_1]. \quad (A.30)
\]

Using the field redefinitions

\[
q_1 = v_1, \quad q_2 = v_2 - v_1, \quad q_3 = -v_2 + v_3 + v_4, \quad q_4 = -v_3 + v_4, \quad (A.31)
\]

we get

\[
L = \frac{1}{4} \left[ \partial + q_1 \right] \left[ \partial + q_2 \right] \left[ \partial + q_3 \right] \left[ \partial + q_4 \right] \left[ \partial - q_1 \right] \left[ \partial - q_2 \right] \left[ \partial - q_3 \right] [\partial - q_4]. \quad (A.32)
\]

which is proportional to the standard result.

We have chosen these two examples because they introduce the techniques needed to generate Lax operators associated with even the most complicated algebraic systems. Furthermore, they show how closely the structure of cyclic affine weight diagrams is linked with the construction of general Lax operators.
As a result, we propose a set of four steps which allows one to construct Lax operators associated with cyclic representations of affine Lie algebras.

First, we propose that to every weight vector of an affine cyclic weight diagram one can associate an operator as follows:

$$Step \ 1 : (a_0a_1a_2 \ldots) \rightarrow [\partial_x + a_1v_1(x) + a_2v_2(x) + \cdots]. \quad (A.33)$$

The coefficient $a_0$ does not appear on the right-hand side as it corresponds to the deleted vertex. Next, we introduce a step which is designed to facilitate the construction of Lax operators when branch points exist in the corresponding weight diagram. Essentially, this step reduces more complicated non-trivial cases to a sum of manageable trivial cases by reducing the branched weight diagram to a sum of linear subdiagrams.

$$Step \ 2: \textit{Replace branching weight diagrams by the sum}$$

of linear subdiagrams, each representing a vertical route \quad (A.34)

beginning with, and ending on, the vacuum weight(s).

For example, Fig. 14 presents the four linear subgraphs associated with the canonical representation of $D_4^{(1)}$.

In drawing cyclic weight diagrams, it is important that the arrows generated by the deleted vertex are distinguished from the others. Our convention is to use dashed lines. Furthermore, the direction of the arrows must also be noted. The Lax operators associated with each subdiagram are then con-
structured as follows:

**Step 3:** Circulate around the loop beginning with the vacuum solution, such that the flow is opposite most of the arrows. If a weight vector is approached by an arrow's

a) tail, append its weight factor to the operator's left side,

b) head, append the weight factor's inverse to the operator's left side

For weights at the tail end of both connecting arrows, do nothing.

Multiply by the product of the corresponding normalization factors.  

\( (A.35) \)

The loop is to be circulated in a direction opposite most of the arrows so that the leading term of the Lax operator \( L = \partial^n + \ldots \) has positive exponent. i.e., \( n > 0 \).

For trivial cases, this completes the computation of \( L \). However, for non-trivial cases with branching weight diagrams we cannot naively build the final Lax operator from a sum of its constituent linear subgraphs. Instead, as we shall prove in the next section, they are added together analogously to how one computes total resistance of resistors in parallel.

**Step 4:** The Lax operator is given by the inverse of the sum of

the inverses, of the operators resulting from step three.  

\( (A.36) \)

For example, in the non-trivial case \( (D_4^{(1)}, c_0) \), with \( \phi = \psi_s \) as before, we build four operators corresponding to the linear subdiagrams in Fig. 14.

\[
\begin{align*}
L_1 &= [\partial - v_1]^{-1}[\partial + v_1][\partial - v_1 + v_2][\partial - v_2 + v_3 + v_4][\partial - v_3 + v_4] \\
& \times [\partial + v_2 - v_3 - v_4][\partial + v_1 - v_2][\partial - v_1]. \\
L_2 &= [\partial - v_1]^{-1}[\partial + v_1][\partial - v_1 + v_2][\partial - v_2 + v_3 + v_4][\partial + v_3 - v_4] \\
& \times [\partial + v_2 - v_3 - v_4][\partial + v_1 - v_2][\partial - v_1]. \\
L_3 &= [\partial - v_1 + v_2][\partial - v_2 + v_3 + v_4][\partial - v_3 + v_4][\partial + v_2 - v_3 - v_3] \\
& \times [\partial + v_1 - v_2][\partial - v_1]. \\
L_4 &= [\partial - v_1 + v_2][\partial - v_2 + v_3 + v_4][\partial - v_3 + v_4][\partial + v_2 - v_3 - v_3] \\
& \times [\partial + v_1 - v_2][\partial - v_1].
\end{align*}
\]

\( (A.37, A.38, A.39) \)
Figure 14. Subdiagram of the cyclic weight diagram of $D_4^{(1)}$. The dashed arrows are the deleted root.

and

$$L_4 = [\partial - v_1 + v_2][\partial - v_2 + v_3 + v_4][\partial + v_3 - v_4][\partial + v_2 - v_3 - v_3]$$

$$\times [\partial + v_1 - v_2][\partial - v_1].$$

Factoring out common terms, we find

$$L^{-1} = \sum_{i=1}^{4} (L_i)^{-1}$$

$$= [\partial - v_1]^{-1}[\partial + v_1 - v_2]^{-1}[\partial + v_2 - v_3 - v_4]^{-1}$$

$$\times \{(\partial - v_3 + v_4)^{-1} + [\partial + v_3 - v_4]^{-1}\}[\partial - v_2 + v_3 + v_4]^{-1}$$

$$\times [\partial - v_1 + v_2]^{-1}[\partial + v_1]^{-1}\{1 + [\partial - v_1]^{-1}[\partial + v_1]\}.$$
By taking the reciprocal, and simplifying, we reproduce the previous result Eq.(A.30).

To end this section we consider the alternative vacuum choice $\psi_3 - \psi_4$. It should be obvious that since we are dealing with cyclic weight diagrams, Lax operators associated with other vacuum states can be achieved by cyclically permuting factors in the primary Lax operator. Thus, this second vacuum choice immediately gives the Lax operator

$$L = \frac{1}{4} [\partial - q_3][\partial - q_2][\partial - q_1][\partial + q_1][\partial + q_2][\partial + q_3][\partial + q_4]^{-1} [\partial - q_4]. \quad \text{(A.42)}$$

Proof of Diagrammatic Scheme.

To prove the equivalence between the matrix system $\mathcal{L} \tilde{\psi} = 0$ and the diagrammatic algorithm, we begin by rewriting the former as

$$\mathcal{D} \tilde{\psi}(x) = -\Lambda \tilde{\psi}(x), \quad \text{(A.43)}$$

where, to simplify notation, we have defined

$$\mathcal{D} \equiv \mathcal{L} - \Lambda = I \frac{\partial}{\partial x} + q(x). \quad \text{(A.44)}$$

The structure of the associated cyclic weight diagram is encoded entirely in the matrix $\Lambda$. Specifically, recall that the general matrix element $\Lambda_{ij}$ is proportional to $\lambda$ if the difference between the $i^{\text{th}}$ weight and the connecting $j^{\text{th}}$ weight equals the eliminated root of the embedding affine simple root system. All other connecting weights $\Lambda_{ij}$ are proportional to 1. Otherwise, the matrix element is assigned the value 0. In all cases the proportionality constant is the normalization factor of the connecting root.

We construct the proof in stages. For the first stage we consider the trivial case, i.e., a single entry in each row and column of $\Lambda$. Removing a root produces
only a single vacuum state. This stage corresponds to weight diagrams with no branch points and only one arrow associated with the eliminated root. Clearly, we can rearrange the matrix equations in \( L \bar{\psi} = 0 \) such that the vacuum state equation appears last. Furthermore, it can be arranged such that \( \Lambda \) is lower triangular with ones located along a diagonal once removed from the main diagonal, except for the eliminated root whose coefficient \( \lambda \) appears in the upper right-hand corner, \( i.e. \), \( \Lambda \) is a circulant matrix. Thus, the Lax eigenvalue equation becomes

\[
L \psi_N(x) = \mu \psi_N(x). \tag{A.45}
\]

Since \( \mathcal{D} \) is diagonal, the \( k^{th} \) equation in (A.43) can be written

\[
\mathcal{D}_k \psi_k = - \sum_{i=1}^{N} \Lambda_{ki} \psi_i, \quad k = 2, 3, \ldots, N. \tag{A.46}
\]

Clearly, since \( \Lambda \) is a circulant matrix as specified above, the inequality \( i < k \) holds for \( k \neq 1 \). Next, by repeatedly replacing the function \( \psi_i \), appearing on the right-hand side, with the \( i^{th} \) matrix equation we eventually reach the expression \( \Lambda_{k1} \psi_1 \), \( i.e. \)

\[
\mathcal{D}_N \psi_N = - \sum_{i=1}^{N} \Lambda_{Ni} \psi_i
\]

\[
= \sum_{i=1}^{N} \sum_{j=1}^{N} \Lambda_{Ni} \mathcal{D}_i^{-1} \Lambda_{ij} \psi_j
\]

\[
= (-)^{\gamma} \sum_{i=1}^{N} \cdots \sum_{k=1}^{N} \Lambda_{Ni} \mathcal{D}_i^{-1} \Lambda_{ij} \cdots \mathcal{D}_k^{-1} \Lambda_{k1} \psi_1
\]

where \( \gamma \) is the height of the cyclic weight diagram. Due to the successive applications of the state lowering operators \( \Lambda_{ij} \) with \( i > j \), this equation is interpreted as taking the highest state \( \psi_1 \) and lowering it to the vacuum state \( \psi_N \).
Replacing \( \psi_1 \) through

\[
D_1 \psi_1 = -N_0 \lambda \psi_N, \tag{A.48}
\]

where we have used \( \Lambda_1 \psi = N_0 \lambda \) which excites the level of the state since \( 1 < N \), gives

\[
D_N \psi_N = -(-)^N N_0 \lambda \sum_{i=1}^{N} \cdots \sum_{k=1}^{N} \Lambda_{N_i} D_i^{-1} \Lambda_{ij} \cdots D_k^{-1} \Lambda_{k1} D_1^{-1} \psi_N. \tag{A.49}
\]

where \( N_0 \) is the normalization factor of the affine root. Finally, moving terms to the left-hand side we recover (A.45) where

\[
L = \{N_0 \sum_{i=1}^{N} \cdots \sum_{k=1}^{N} D_N^{-1} \Lambda_{N_i} D_i^{-1} \cdots \Lambda_{k1} D_1^{-1}\}^{-1}. \tag{A.50}
\]

and the spectral parameter is given by

\[
\mu = -(-)^N \lambda. \tag{A.51}
\]

Since each row and column of \( \Lambda \) contain only one entry the sum will generate a single term, \( i.e., \)

\[
L^{-1} = \{N \Lambda D_N^{-1} D_{N-1}^{-1} \cdots D_1^{-1}\}^{-1}, \tag{A.52}
\]

where \( N \) is the product of the normalization factors. Now, each \( D \) is a weight factor as defined in step 1. Thus, there is a direct mapping between the order of the weight factors and their location in the corresponding weight diagram.

Now, suppose we permit multiple row entries in \( \Lambda \), \( i.e., \) branch points in the weight diagram. First consider the case where such multiple entries occur above the \( N^{th} \) row. As before, there is a single vacuum state, and the constant \( \lambda \) is located in the upper right-hand corner of \( \Lambda \). Therefore, the constraint \( i < k \), for \( k \neq 1 \), remains in effect for Eq.(A.46). Hence, the derivation leading to (A.50) follows through unchanged. Now, each new entry in \( \Lambda \) causes an additional final term in (A.50). Clearly, \( per \) step 4 of the diagrammatic algorithm, the
final Lax operator is obtained by taking the reciprocal of the sum of terms generated by (A.50).

Next, suppose the multiple row entries in $\Lambda$, due to the branch point, occur in the $N^{th}$ row. The vacuum condition (A.10) shows that this is equivalent to a degenerate vacuum state with, say, degeneracy $d$. Subsequently, this row will be associated with the eliminated root, and the $d$ integers will be assigned the value $\lambda$. In fact, $\lambda$ appears only in this row. Clearly, in the weight diagram the $d$ weights share the same level.

Let us first discuss the case where the coefficients $\lambda$ occur in the first row. Thus, Eq.(A.46) remains valid, keeping intact the constraint $i < k$, for $k \neq 1$. Furthermore, the scalar eigenfunction $\phi(x)$ is now a linear combination of the components $\psi_N, \psi_{N-1}, \ldots, \psi_{N-d+1}$, and the equation for $\psi_1$ becomes

$$D_1 \psi_1 = - \sum_{s=N-d+1}^{N} \Lambda_s \psi_s.$$  \hspace{1cm}(A.53)

Consequently, Eq.(A.49) is modified to

$$\psi_s = -(-)^{\beta} \sum_{i=1}^{N} \cdots \sum_{k=1}^{N} \sum_{j=N-d+1}^{N} D_s^{-1} \Lambda_{si} D_i^{-1} \cdots$$

$$\times D_k^{-1} \Lambda_{k1} D_1^{-1} \Lambda_{1j} \psi_j,$$

where $\beta$ is the number of field replacements performed. The characteristic equation is obtained by multiplying both sides by $\Lambda_i$, and then summing over $s$, i.e.,

$$\sum_{s=N-d+1}^{N} \Lambda_s \psi_s = -(-)^{\beta} \sum_{s=1}^{N} \cdots \sum_{j=N-d+1}^{N} \Lambda_s D_s^{-1} \Lambda_{si} \cdots$$

$$\times D_k^{-1} \Lambda_{k1} D_1^{-1} \Lambda_{1j} \psi_j.$$  \hspace{1cm}(A.55)

Note, the sum over $s$ on the right-hand side has been extended to the entire range for convenience.
Each term in \( \Lambda_i \) contains the factor \( \lambda \), which can then be factored out. As a result, the scalar eigenfunction is found to be

\[
\phi = \sum_{s=N-d+1}^{N} \Lambda_i \psi_s, \tag{A.56}
\]

and the Lax operator,

\[
L = \lambda \left\{ \prod_{s=1}^{N} \Lambda_s^{-1} \Lambda_{s_1} \cdots \Lambda_{s_k} \right\}^{-1}. \tag{A.57}
\]

Clearly, this has the same interpretation as the non-degenerate branching case.

For the last stage of the proof, we relax the condition that multiple occurrences of \( \lambda \) must all be in the first row of \( \Lambda \). In the weight diagram this means not all the arrows associated with the eliminated root point to the bottom level. Recall from the discussion surrounding Eq.(A.9), the eliminated root with coefficient \( \lambda \) acts as a state raising operator. Thus, every occurrence of \( \lambda \) will appear in the upper triangular portion of \( \Lambda \), and the unit coefficients of the state lowering roots are in the lower triangular portion.

For \( \lambda \) in the \( k^{th} \) row of \( \Lambda \), \( k \neq 1 \), Eq.(A.46) is modified to

\[
D_j \psi_j = -\lambda \psi_k - \sum_{n \neq k}^{N} \Delta_{-jn} \psi_n, \tag{A.58}
\]

where, since \( \lambda \) corresponds to the state raising operator, \( k > j \). Consider the case where \( j \) is the largest such index to satisfy this equation. Then, allowing degenerate vacuum states, we have

\[
\psi_s = (-)^{\beta} \sum_{i=1}^{N} \ldots \sum_{j=1}^{N} \Lambda_{s_i}^{-1} \Lambda_{s_1}^{-1} \cdots \Lambda_{s_k} \psi_j \tag{A.59}
\]

where, again, \( \beta \) is the number of field replacements performed. The effect of the factor in front of \( \psi_k \) is to first, due to \( \lambda \), raise this state to \( \psi_j \) and then
to lower it till the vacuum state $\psi$, is reached. Now, since the corresponding weight diagram is cyclic, there must exist some factor that will circulate $\psi$, back to $\psi_k$.

First, as was the case with $(D_4^{(1)}, e_0)$, consider the situation where $\psi_k$ is an intermediate state in (A.59), i.e.,

$$\psi' = (-)^{\alpha} \sum_{i=1}^{N} \sum_{k=1}^{N} D_{s}^{-1} \Lambda_{si} D_{k}^{-1} \cdots \Lambda_{jk} \psi_k.$$  \hspace{1cm} (A.60)

This gives

$$\psi_k = (-)^{\alpha} \left\{ \sum_{i=1}^{N} \sum_{k=1}^{N} D_{s}^{-1} \Lambda_{si} D_{k}^{-1} \cdots \Lambda_{jk} \right\}^{-1} \psi'.$$  \hspace{1cm} (A.61)

Thus, per step 3 of the diagrammatic algorithm, the factor $D_k$, associated with the weight vector at the tail end of both connecting arrows does not appear.

Further, proceeding from higher weights to lower weights in the weight diagram contributes factors of $D_k^{-1}$ in the operator defined in step 2 for the linear subdiagram.

Finally, if $\psi_k$ does not appear as an intermediate state of the vacuum state $\psi$, in (A.59), then it must occur as an intermediate state for one of the other vacuum states. Again, since the weight diagram is cyclic, there is some closed path going from $\psi$, to each of these other vacua. However, to write down a final expression is too unwielding. Nevertheless, it should be clear that the general rules of the diagrammatic algorithm are complete and provide an accurate mapping between weight diagrams and the scalar Lax equation.

**Supersymmetric Lax Operators**

In this section we generalize the diagrammatic scheme to supersymmetric affine Lie algebras. The classification of all possible supersymmetric extensions of the basic Lie algebras has been given by Kac.\(^{48}\) In addition to the
bosonic simple roots of the basic Lie algebra, the simple root system of the
supersymmetric algebras contains two distinct kinds of fermionic roots. The
Dynkin symbol of the first type is sometimes given by a shaded vertex repre-
senting a non-zero norm. The second fermionic root type has zero norm whose
Dynkin symbol is given correspondingly by a crossed out vertex. As always,
the bosonic root is denoted by a white vertex.

A new feature occurring in the supersymmetric Lie algebras is that they
may have several non-equivalent simple root systems, corresponding to differ-
et Dynkin diagrams and Cartan matrices. In other words, the different root
systems can not be transformed into each other through standard Weyl rota-
tions. Instead, they are obtained by performing the “Weyl” transformation
with respect to the nilpotent fermionic root. For more details, see Frappat et
al. which also presents a large collection of Dynkin diagrams associated with
all of the classical contragradient supersymmetric cases, those of the affine and
twisted affine supersymmetric algebras.

Non-equivalent simple root systems which represent the same supersym-
metric Lie algebra differ in the distribution of bosonic and fermionic roots.
However, here we are interested in considering a natural extension of the
Drinfel’d-Sokolov procedure to the supersymmetric case. This restricts the
possible choices for the simple root system used for building the supersym-
metric Lax operators. Recall, in the bosonic case the mKdV Lax operator
constructed with the gradation choice of Drinfel’d and Sokolov generates Toda
lattice models. For supersymmetric algebras it has been shown that
Toda lattices are possible only for simple root systems composed purely of
fermionic roots. Supersymmetric Lie algebras with purely fermionic root sys-
tems have been given by Leites et al.\textsuperscript{54}

\[ SL(n+1 \mid n), OSp(m \mid 2n) \ (m = 2n, 2n + 2, 2n \pm 1), D(2 \mid 1; \alpha). \] \hfill (A.62)

Furthermore, the infinite-dimensional affine supersymmetric Lie algebras with purely fermionic simple root systems are

\[ SL(n \mid n)^{(1)}, OSp(2n+2 \mid 2n)^{(1)}, D(2 \mid 1; \alpha)^{(1)}, \] \hfill (A.63)

while the infinite-dimensional twisted affine cases are

\[ SQ(2n+1)^{(2)}, SL(n \mid n)^{(2)}, OSp(2n \mid 2n)^{(2)}. \] \hfill (A.64)

The supersymmetric extension of the KdV equations was first discussed in Manin and Radul.\textsuperscript{55} They suggested replacing the bosonic derivative \( \partial_x \) by its supersymmetric analog, \( i.e., \)

\[ \partial_x \rightarrow D = \frac{\partial}{\partial \theta} + \theta \frac{\partial}{\partial x}. \] \hfill (A.65)

Note that \( D^2 = \frac{\partial}{\partial x} \). The system of matrix equations of Drinfel’d and Sokolov can then be generalized to\textsuperscript{50}

\[ \mathcal{L} \tilde{\psi}(x, \theta) \equiv [D + Q(x, \theta) + \Lambda] \tilde{\psi}(x, \theta) = 0, \] \hfill (A.66)

where \( \Lambda \) is generated by the purely negative fermionic roots, and \( Q(x, \theta) \) is a Grassmann odd fermionic superfield which can be expanded as

\[ Q(x, \theta) = \sum_{i=1}^{r} H_i \Psi_i(x, \theta), \] \hfill (A.67)

where now \( H_i \) are elements of the Cartan-Kac subalgebra. The vacuum condition is as before,

\[ \Delta^- \tilde{\psi}(x, \theta) = 0. \] \hfill (A.68)
Since the second type of fermionic root is nilpotent, they deserve special treatment when constructing cyclic weight diagrams. To illustrate how this comes about, consider the fundamental representation of the supersymmetric algebra $OSp(2 \mid 2)$. The Dynkin diagram of the purely fermionic root system is given by

\[
\begin{array}{c}
\times \quad 1 \\
\times \\
\end{array}
\]

where both fermionic roots are denoted as having zero norm, and where we have indicated the choice (11) for a highest weight vector. The Cartan matrix is then easily found to be

\[
A = \begin{pmatrix}
0 & -2 \\
-2 & 0
\end{pmatrix}.
\]  

(A.69)

To construct the highest weight diagram we proceed as before. Since a positive one appears both in the first and second places we have two permutations of subtraction to perform. In particular, we can start by subtracting the first row of the Cartan matrix giving (13), and then subtracting the second row resulting in (33). However, unlike the bosonic case, we may not subtract the first row of the Cartan another time from the weight (13). This is because here the fermionic weight vectors are nilpotent and subtracting any Cartan row twice gives a decoupled state. Similarly, we can start by subtracting the second Cartan row once (and only once) and then the first row giving (33). Thus, we find the weight diagram with height two:

\[
(11)_1 \quad (13)_2 \quad (31)_3 \quad (33)_4
\]

(A.70)

The decoupling which occurs when constructing a cyclic weight diagram for an affine supersymmetric algebra is almost as straightforward. For example, in Fig. 15 we display the partially decoupled weight diagram of $(SL(2 \mid 2)^{(1)}, c_0)$
where states were decoupled as we went from top to bottom. There are several ways to decouple the remaining weights since the lowering operators $b_0, b_2$ and $b_3$ still appear more than once. The only way for a cyclic weight diagram to emerge is by decoupling the weights outside the box. To see that this is also consistent, note that all paths leading from weight 5 to weights 2 or 4 require two applications of $b_0$.

To construct a super-Lax operator let us take the vacuum solution $\phi = \Psi_4$. We easily find the super-Lax operator to be

$$L = [D + \Psi_3][D + \Psi_2 + \Psi_3][D + \Psi_1 + \Psi_2][D + \Psi_1].$$

(A.71)
As in the non-supersymmetric case, Lax operators corresponding to the other three vacua are obtained through cyclic permutations of the above operator.

**Discussion**

In this appendix we have shown how one can read off from cyclic weight diagrams, associated with representations of affine Lie algebras, Lax operators in the diagonal gauge. This method is most useful when tables of matrix representations are not at hand and must be generated by weight diagrams anyway. Furthermore, this procedure can easily be implemented on computer by virtue of the fact that computer generated algorithms currently exist for building highest weight diagrams.56 With minor modifications, these programs can be adapted for cyclic weight diagrams.

It remains to be seen whether higher representations lead to any new physics. If so then a program of categorizing these results might be pursued to identify redundant solutions. This might be easier to answer for supersymmetric algebras since nilpotency projects out decoupled weight vectors. What is clear though, at least for the non-supersymmetric cases is that these higher representations lead to integrable systems. Recall, to prove the integrability of KdV systems Drinfel’d and Sokolov found the necessary infinite set of conserved currents to be given by the coefficients of the Laurent expansion of $\mathcal{L}$ in the affine parameter $\lambda$. Our conclusion follows from the fact that every representation of a basic Lie algebra has an affine extension, and that defining properties of affine Lie algebras are representation independent.

Finally, it would be interesting to see if our procedure could be modified to directly generate Lax operators in other gauges. Furthermore, in light of recent work47 on generalizations of the Drinfel’d and Sokolov scheme, one may
also consider different gradations of the affine Lie algebra from which to obtain the matrix Λ and the form of q(x).