1 Introduction

The last seven years have been a very exciting time for string theory. A new understanding of nonperturbative objects in string theory, such as D-branes, has led to exciting new developments that relate string theory to physical systems such as black holes and supersymmetric gauge field theories. It has also led to the discovery of unexpected relationships between Yang-Mills theories and quantum theories of gravity such as closed superstring theories and M-theory. Finally, the analysis of unstable D-branes has elucidated the long-standing mysteries associated with the open string tachyon.

The study of unstable D-branes and tachyons has also led to the realization that string field theory contains significant non-perturbative information. This has been somewhat of a surprise. Certain forms of string field theory were known since the early 1990's, but there was no concrete evidence that they could be used to give a non-perturbative definition of string theory. The study of tachyon condensation, however, has changed our perspective. These lecture notes give an introduction to string field theory and review recent work in which unstable D-branes and their associated tachyons are described using string field theory. As we will discuss here, this work suggests that open string field theory, or some successor of it, may give a complete definition of string theory in which all possible backgrounds can be obtained from a single set of degrees of freedom. Such a formulation appears to be necessary to address questions related to vacuum selection and string cosmology.

In the rest of this section we will review briefly the current status of string theory as a whole, and summarize the goals of this set of lectures. In section 2 we review some basic aspects of D-branes. In section 3, we describe a particular D-brane configuration which exhibits a tachyonic instability. This tachyon can be seen in the low-energy Yang-Mills description of the D-brane system. We also describe a set of conjectures made by Sen in 1999, which stated that the tachyonic instability of the open bosonic string is the instability of the space-filling D25-brane. Sen suggested that open string field theory could be used to give an analytic description of this instability. In section 4 we give an introduction to Witten's bosonic open string field theory (OSFT). Section 5 gives a more detailed analytic description of this theory using the language of conformal field theory. Section 6 describes the string field theory using the oscillator approach and overlap integrals. The two approaches to OSFT described in these two sections give complementary ways of analyzing problems in string field theory. In section 7 we summarize evidence from string field theory for Sen's conjectures. In section 8 we describe "vacuum string field theory," a new version of open string field theory which arises when one attempts to directly formulate the theory around the classically stable vacuum where the D-brane has disappeared. This section also discusses important structures in string field theory, such as projectors of the star algebra of open string fields. Section 9 contains concluding remarks.

Much new work has been done in this area since these lectures were presented at TASI in 2001. Except for some references to more recent developments which are related to the topics covered, these lecture notes primarily cover work done before summer of 2001. Previous articles reviewing related work include those of Ohmori [1], de Smet [2], Aref'eva *et al.* [3], Bonora *et al.* [4], and Taylor [5]. There are a number of major related areas which we do not cover significantly or at all in these lectures. We do not have any substantial discussion on the dynamic process of tachyon decay; there has been quite a bit of work on this subject [6] since the time of these lectures in 2001. We do not discuss the Moyal approach to SFT taken recently by Bars and collaborators [7, 8, 9]; this work is an interesting alternative to the level-truncation method primarily used here. We also do not discuss in any detail the alternative boundary string field theory (BSFT) approach to OSFT. The BSFT approach is well suited to derive certain concrete results regarding the tachyon vacuum [10]—for example, using this approach the energy of the tachyon vacuum can be computed exactly. On the other hand, BSFT is not a completely well-defined framework, as massive string fields cannot yet be consistently incorporated into the theory.

1.1 The status of string theory: a brief review

To understand the significance of developments over the last seven years, it is useful to recall the status of string theory in early 1995. At that time it was clearly understood that there were five distinct ways in which a supersymmetric string could be quantized to give a microscopic definition of a theory of quantum gravity in ten dimensions. Each of these quantum string theories gives a set of rules for calculating scattering amplitudes of string states; these states describe gravitational quanta and other massless and massive particles moving in a ten-dimensional spacetime. The five superstring theories are known as the type IIA, IIB, I, heterotic $SO(32)$, and heterotic $E_8 \times E_8$ theories. While these string theories give perturbative descriptions of quantum gravity, there was little understanding in 1995 of nonperturbative aspects of these theories.

In the years between 1995 and 2002, several new ideas dramatically transformed our understanding of string theory. We now briefly summarize these ideas and mention some aspects of these developments relevant to the main topic of these lectures.

Dualities: The five different perturbative formulations of superstring theory are all related to one another through duality symmetries [11, 12], whereby the degrees of freedom in one theory can be described through a duality transformation in terms of the degrees of freedom of another theory. Some of these duality symmetries are nonperturbative, in the sense that the string coupling g in one theory is related to the inverse string coupling $1/g$ in the dual theory. The web of dualities that relate the different theories gives a picture in which, rather than describing five distinct fundamental theories, each superstring theory appears to be a particular perturbative limit of a single, still unknown, underlying theoretical structure.

M-theory: In addition to the five perturbative string theories, the web of dualities also seems to include a limit which describes a quantum theory of gravity in eleven dimensions. This new theory has been dubbed "M-theory". Although no covariant definition for M-theory has been given, this theory can be related to type IIA and heterotic $E_8 \times E_8$ string theories through compactification on a circle S^1 and the space S^1/Z_2, respectively [13, 12, 14]. In the relation to type IIA, for example, the compactification radius R_{11} of M-theory is equal to the product $g_s l_s$ of the string coupling g_s and the string length l_s. Thus, M-theory in flat space, which arises in the limit $R_{11} \to \infty$, can be thought of as the strong coupling limit of type IIA string theory. The field theory limit of M-theory is eleven-dimensional supergravity. It is also suspected that M-theory may be formulated as a quantum theory of membranes in eleven dimensions [13].

Branes: In addition to strings, all five superstring theories, as well as M-theory, contain extended objects of various dimensionalities known as "branes". M-theory has M2-branes and M5-branes, which have two and five dimensions of spatial extent, respectively. (A string is a one-brane, since it has one spatial dimension.) The different superstring theories each have different sets of (stable) D-branes, special branes that are defined by Dirichlet-type boundary conditions on strings. In particular, the IIA/IIB superstring theories contain (stable) D-branes of all even/odd dimensions. Each superstring theory also has a fundamental string and a Neveu-Schwarz five-brane. The branes of one theory can be related to the branes of another through the duality transformations mentioned above. Using an appropriate sequence of dualities, any brane can be mapped to any other brane, including the string itself. This suggests that none of these objects are really any more fundamental than any others; this idea is known as "brane democracy".

M(atrix) theory and AdS/CFT: It is a remarkable consequence of the above developments that for certain asymptotic space-time backgrounds, M-theory and string theory can be completely described through supersymmetric quantum mechanics and field theories related to the low-energy description of systems of branes. The M(atrix) model of M-theory is a simple supersymmetric matrix quantum mechanics, and it is believed to capture (in light-cone coordinates) all of the physics of M-theory in asymptotically flat spacetime. In the AdS/CFT correspondence, certain maximally supersymmetric Yang-Mills theories can be used to describe closed superstring theories in asymptotic spacetime backgrounds that are the product of anti-de Sitter space and a sphere. It is believed that the Yang-Mills theories and the matrix model of M-theory, each give true nonperturbative descriptions of quantum gravity in the corresponding spacetime geometry. For reviews of M(atrix) theory and AdS/CFT, see Taylor [15] and Aharony*et.al* [16].

Unstable D-branes and open string tachyons: This is in large part the subject of these lectures. The most recent chapter in our new understanding of nonperturbative effects in string theory has been the incorporation of unstable branes and open string tachyons into the overall framework of the theory. It has turned out that an understanding of unstable D-branes is necessary to properly describe all D-branes. This is natural from the point of view of K-theory, where brane configurations which are equivalent under the annihilation of unstable branes are identified [17]. The long-mysterious tachyon instability of open string theory has finally been given a physical interpretation: it is the instability of the D-brane that supports the existence of open strings. The instability disappears in the tachyon vacuum, in which the D-brane decays. Moreover, the belief that D-branes are solitonic solutions of string theory has been confirmed: starting with the appropriate tachyonic field theory of unstable space-filling branes, one can describe lower dimensional D-branes as solitonic solutions. Lower dimensional D-branes are thereby essentially obtained as solitons of the tachyon field theory, so, in some sense, lower-dimensional D-branes can be thought of as being made of tachyons! It has also been shown that the physics of unstable D-branes is captured by string field theory, thus making it a candidate for a non-perturbative formulation of string theory capable of describing changes of the string background.

The set of ideas just summarized have greatly increased our understanding of nonperturbative aspects of string theory. In particular M(atrix) theory and the AdS/CFT correspondences provide nonperturbative definitions of M-theory and string theory in certain asymptotic space-time backgrounds which can be used, in principle, to calculate any local result in quantum gravity. Through string field theory we have a possibly nonperturbative definition of the theory that appears to capture many open string theory backgrounds. The existing formulations of string field theory are not manifestly background independent because a background must be selected to write the theory. Nevertheless, as we discuss in these lectures, the theory describes multiple distinct backgrounds in terms of a common set of variables, so it embodies, at least partially, physical background independence. It remains to be seen if the theory incorporates full physical background independence; this requires an ability to describe all possible open string backgrounds, as well as all possible closed string backgrounds.

1.2 The goal of these lectures

The goal of these lectures is to describe progress towards a nonperturbative formulation of string theory that implements the physics of background independence. Open string field theory, as applied to tachyon condensation and related matters, has shown itself capable of describing non-perturbative objects in string theory, and it has demonstrated an ability to represent various open string backgrounds.

A completely background independent formulation of string theory may be needed to address fundamental questions such as: What is string theory/M-theory? How is the vacuum of string theory selected? (*i.e.*, Why can the observable low-energy universe be accurately described by the standard model of particle physics in four space-time dimensions with an apparently small but nonzero positive cosmological constant?), and other questions of a cosmological nature. Obviously, aspiring to address these questions is an ambitious undertaking, but we believe that attaining a better understanding of string field theory is a useful step in this direction. More concretely, in these lectures we will describe recent progress on open string field theory. It may be useful here to recall some basic aspects of open and closed strings and the relationship between them.

Closed strings, which are topologically equivalent to a circle S^1, give rise upon quantization to a massless set of states associated with the graviton $g_{\mu\nu}$, the dilaton φ, and the antisymmetric two-form $B_{\mu\nu}$, as well as an infinite family of massive states. For the supersymmetric closed string, further massless fields appear within the graviton supermultiplet—these are the Ramond-Ramond p-form fields $A^{(p)}_{\mu_1 \cdots \mu_p}$ and the gravitini $\psi_{\mu\alpha}$. Thus, the quantum theory of closed strings is naturally associated with a theory of gravity in space-time. On the other hand, open strings, which are topologically equivalent to an interval $[0, \pi]$, give rise under quantization to a massless gauge field A_μ in space-time. The supersymmetric open string also has a massless gaugino field ψ_α. It is now understood that the endpoints of open strings must lie on a Dirichlet p-brane (Dp-brane), and that the massless open string fields describe the fluctuations of the D-brane and the gauge field living on the world-volume of the D-brane.

It may seem, therefore, that open and closed strings are quite distinct, and describe disjoint aspects of the physics in a fixed background space-time that contains some family of D-branes. At tree level, the closed strings indeed describe gravitational physics in the bulk space-time, while the open strings describe the D-brane dynamics. At the quantum level, however, the physics of open and closed strings are deeply connected. Indeed, historically open strings were discovered first through the form of their scattering amplitudes [18]. Looking at one-loop processes for open strings led to the first discovery of closed strings, which appeared as *poles* in nonplanar one-loop open string diagrams [19, 20]. The fact that open string diagrams naturally contain closed string intermediate states indicates that in some sense all closed string interactions are implicitly defined by the open string diagrams. This connection underlies many of the important recent developments in string theory. In particular, the M(atrix) theory and AdS/CFT correspondences between gauge theories and quantum gravity are essentially limits in which closed string physics in a fixed space-time background is captured by the Yang-Mills limit of an open string theory on a family of branes (D0-branes for M(atrix) theory, D3-branes for the CFT that describes $\mathrm{AdS}_5 \times S^5$, etc.)

Since quantum gravity theories in certain fixed space-time backgrounds can be described by field theory limits of open strings, we may ask if a global change of the space-time background can be described as well. If M(atrix) theory or AdS/CFT allowed for this description, it would indicate that these models may have background-independent generalizations. Unfortunately, such background changes involve the generally intractable addition of an infinite number of nonrenormalizable interactions to the field theories in question. One tractable situation arises for the addition of a constant background $B_{\mu\nu}$ field in space-time (perhaps because this closed string background is gauge equivalent to the open string background of a D-brane with a magnetic field). In the associated Yang-Mills theory, this change in the background field corresponds to replacing products of open string fields with a noncommutative star-product. The resulting theory is a noncommutative Yang-Mills theory. Such noncommutative theories are the only well-understood example of a situation where adding an infinite number of apparently nonrenormalizable terms to a field theory action leads to a sensible modification of quantum field theory (for a review of noncommutative field theory and its connection to string theory, see Douglas and Nekrasov [21]).

String field theory is a nonperturbative formulation of string theory in which the infinite family of fields associated with string excitations are described by a space-time field theory action. For open strings on a D-brane configuration, this field theory contains Yang-Mills fields and an entire hierarchy of massive string fields. Integrating out all the massive fields from the string field theory action gives rise to a nonabelian Born-Infeld action for the D-branes, which includes an infinite set of higher-order terms that arise from string theory corrections to the simple Yang-Mills action. Like the case of noncommutative field theory discussed above, the new terms appearing in this action are apparently nonrenormalizable, but the combination of terms must work together to form a sensible theory.

In the 1980's, a great deal of work was done to formulate string field theory for open and closed, bosonic and supersymmetric string theories. All of this work was based on the BRST approach to string quantization [22, 23, 24, 25]. For the open bosonic string Witten [26] constructed an extremely elegant string field theory based on the Chern-Simons action. This cubic open string field theory (OSFT) is the primary focus of the work described in these lectures. Although this theory can be described in a simple abstract language, practical computations rapidly become complicated. The formulation of bosonic closed string field theory was completed in the early 1990s [27, 28, 29, 30]. This theory is the natural counterpart of Witten's open string field theory, but it is more technically challenging because of its nonpolynomiality. A nonpolynomial string field theory is also required to describe in a non-singular fashion open and closed string fields [31]. For open superstrings, a cubic formulation [32] encountered some difficulties [33, 34] (for which there are some proposed resolutions [35, 36]), but the nonpolynomial formulation of Berkovits [37] appears to be fully consistent. Despite a substantial amount of work in string field theory in the early 90's, little insight was gained at the time concerning non-perturbative physics. Work on this subject stalled out until open string field theory was used to test the tachyon conjectures beginning in 1999 [38].

One simple feature of the 26-dimensional bosonic string has been problematic since the early days of string theory: both the open and closed bosonic strings have tachyons in their spectra, indicating that the usual perturbative vacua of these theories are unstable. In 1999, Ashoke Sen had a remarkable insight into the nature of the open bosonic string tachyon [39]. He observed that the open bosonic string theory (the so-called Veneziano model) represents open strings that end on a space-filling D25-brane. He pointed out that this D-brane is unstable, as it does not carry any conserved charge, and he suggested that the open string tachyon is in fact the unstable mode of the D25-brane. This led him to conjecture that open string field theory could be used to precisely determine a new vacuum for the open string, namely one in which the D25-brane is annihilated through condensation of the tachyonic unstable mode. Sen made several precise conjectures regarding the details of the string field theory description of this new open string vacuum. As we describe in these lectures, there is now overwhelming evidence that Sen's picture is correct, demonstrating that string field theory accurately describes the nonperturbative physics of D-branes. This new nonperturbative application of string field theory has sparked a new wave of work on open string field theory, revealing many remarkable new structures. In particular, string field theory now provides a concrete framework in which disconnected string backgrounds can emerge from the equations of motion of a single underlying theory. Although so far this can only be shown explicitly in the open string context, this work paves the way for a deeper understanding of background-independence in quantum theories of gravity.

2 D-branes

In this section we briefly review some basic features of D-branes. The concepts developed here will be useful to describe tachyonic D-brane configurations in the following section. For more detailed reviews of D-branes, see the reviews of Polchinski [40], and of Taylor [41].

2.1 D-branes and Ramond-Ramond charges

D-branes can be understood from many points of view. In these lectures we primarily focus on the viewpoint motivated by the recent work on tachyon condensation, which is that that D-branes are solitons in string field theory. The original realization of the importance of D-branes in string theory stemmed from Polchinski's realization that D-branes could be described in two complementary fashions: *a*) as extended extremal black brane solutions of supergravity that carry conserved charges, and *b*) as hypersurfaces on which strings have Dirichlet boundary conditions. We now discuss these two viewpoints briefly.

a) The ten-dimensional type IIA and IIB supergravity theories each have a set of $(p+1)$-form fields $A^{(p+1)}_{\mu_1\cdots\mu_{(p+1)}}$ in the supergraviton multiplet, with p even/odd for type IIA/IIB supergravity. These are the Ramond-Ramond (RR) fields in the massless superstring spectrum. For each of these $(p+1)$-form fields, there is a solution of the supergravity field equations that is invariant under $(p+1)$-dimensional Lorentz transformations, and which has the form of an extremal black hole solution in the $9-p$ spatial directions that are not affected by these Lorentz transformations [42]. These "black p-brane" solutions carry charge under the RR fields $A^{(p+1)}$, and are BPS states in the supergravity theory that preserve half the supersymmetry of the theory. These solutions represent the gravitational and gauge backgrounds created by the branes, in a way similar to that in which the Schwarzschild solution represents the gravitational background of a point mass, or the Coulomb field represents the electric field of a point charge.

b) In type IIA and IIB string theory, it is possible to consider open strings with Dirichlet boundary conditions on some number $9-p$ of the spatial coordinates $x^\mu(\sigma)$. The locus of points defined by such Dirichlet boundary conditions defines a $(p+1)$-dimensional hypersurface Σ_{p+1} in the ten-dimensional spacetime. When p is even/odd in type IIA/IIB string theory, the spectrum of the resulting quantum open string theory contains a massless set of fields $A_\alpha, \alpha = 0, 1, \ldots, p$ and $X^a, a = p+1, \ldots, 9$. These fields can be associated with a gauge field living on the hypersurface Σ_{p+1}, and a set of degrees of freedom describing the transverse fluctuations of this hypersurface in spacetime, respectively. Thus, the quantum fluctuations of the open string describe a fluctuating $(p+1)$-dimensional hypersurface in spacetime — a Dirichlet-brane, or "D-brane".

The remarkable insight of Polchinski [43] in 1995 was the observation that the stable

Dirichlet-branes of superstring theory carry Ramond-Ramond charges, and therefore should be described in the low-energy supergravity limit of string theory by precisely the black p-branes discussed in *a*). This connection between the string and supergravity descriptions of these nonperturbative objects paved the way to a dramatic series of new developments in string theory, including connections between string theory and supersymmetric gauge theories, string constructions of black holes, and new approaches to string phenomenology. The bosonic D-branes on which we concentrate attention in these lectures do not carry conserved charges, and thus are not associated with supergravity solutions as in *a)*; rather, these D-branes can be described through open bosonic strings with some Dirichlet boundary conditions as in *b)*.

2.2 Born-Infeld and super Yang-Mills D-brane actions

In this subsection we briefly review the low-energy super Yang-Mills description of the dynamics of one or more D-branes. As discussed in the previous subsection, the massless open string modes on a Dp-brane in type IIA or IIB superstring theory describe a $(p+1)$-component gauge field A_α, $9-p$ transverse scalar fields X^a, and a set of massless fermionic gaugino fields. The scalar fields X^a describe small fluctuations of the D-brane around a flat hypersurface. If the D-brane geometry is sufficiently far from flat, it is useful to describe the D-brane configuration by a general embedding $X^\mu(\xi)$, where ξ^α are $p+1$ coordinates on the Dp-brane world-volume $\Sigma_{(p+1)}$, and X^μ are ten functions giving a map from $\Sigma_{(p+1)}$ into the space-time manifold $\mathbf{R}^{9,1}$. Just as the Einstein equations which govern the geometry of spacetime arise from the condition that the one-loop contribution to the closed string beta function vanishes, a set of equations of motion for a general Dp-brane geometry and associated world-volume gauge field can be derived from a calculation of the one-loop open string beta function [44]. These equations of motion arise from the classical Born-Infeld action:

$$S = -T_p \int d^{p+1}\xi \; e^{-\varphi} \sqrt{-\det(G_{\alpha\beta} + B_{\alpha\beta} + 2\pi\alpha' F_{\alpha\beta})} + S_{\rm CS} + \text{fermions} \tag{1}$$

where G, B, and φ are the pullbacks of the ten-dimensional metric, antisymmetric tensor, and dilaton to the D-brane world-volume, while F is the field strength of the world-volume $U(1)$ gauge field A_α. $S_{\rm CS}$ represents a set of Chern-Simons terms which will be discussed in the following subsection. This action can be verified by a perturbative string calculation [40], which also gives a precise expression for the brane tension

$$\tau_p = \frac{T_p}{g_s} = \frac{1}{g_s\sqrt{\alpha'}} \frac{1}{(2\pi\sqrt{\alpha'})^p} \tag{2}$$

where $g_s = e^{\langle\varphi\rangle}$ is the closed string coupling, equal to the exponential of the dilaton expectation value.

A particular limit of the Born-Infeld action (1) is useful to describe many low-energy aspects of D-brane dynamics. Take the background space-time $G_{\mu\nu} = \eta_{\mu\nu}$ to be flat, and all other supergravity fields $(B_{\mu\nu}, A^{(p+1)}_{\mu_1\cdots\mu_{p+1}})$ to vanish. We then assume that the D-brane is approximately flat, and is close to the hypersurface $X^a = 0, a > p$, so that we may make the static gauge choice $X^\alpha = \xi^\alpha$. We furthermore take the low-energy limit in which $\partial_\alpha X^a$ and $2\pi\alpha' F_{\alpha\beta}$ are small and of the same order. The action (1) can then be expanded as

$$S = -\tau_p V_p - \frac{1}{4g_{\rm YM}^2}\int d^{p+1}\xi \left(F_{\alpha\beta}F^{\alpha\beta} + \frac{2}{(2\pi\alpha')^2}\partial_\alpha X^a \partial^\alpha X^a\right) + \cdots \tag{3}$$

where V_p is the p-brane world-volume and the coupling $g_{\rm YM}$ is given by

$$g_{\rm YM}^2 = \frac{1}{4\pi^2\alpha'^2\tau_p} = \frac{g_s}{\sqrt{\alpha'}}(2\pi\sqrt{\alpha'})^{p-2}\,. \tag{4}$$

Ignoring fermionic terms, the second term in the right-hand side of (3) is simply the reduction to $(p+1)$ dimensions of the ten-dimensional $\mathcal{N} = 1$ super Yang-Mills action:

$$S = \frac{1}{g_{\rm YM}^2}\int d^{10}\xi \,\left(-\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{i}{2}\bar{\psi}\Gamma^\mu\partial_\mu\psi\right) \tag{5}$$

where for $\alpha, \beta \leq p$, $F_{\alpha\beta}$ is the world-volume $U(1)$ field strength, and for $a > p, \alpha \leq p$, $F_{\alpha a} \rightarrow \partial_\alpha X^a/(2\pi\alpha')$.

When multiple Dp-branes are present, the D-brane action is modified in a fairly simple fashion [45]. Consider a system of N coincident D-branes. For every pair of branes $\{i, j\}$ there is a set of massless fields

$$(A_\alpha)_i^{\;j}, \quad (X^a)_i^{\;j}\,, \tag{6}$$

associated with strings stretching from the ith brane to the jth brane; the indices i, j are known as Chan-Paton indices. Treating the fields (6) as N-by-N matrices, and letting Tr denote the trace operation for such matrices, the multiple brane analogue of the Born-Infeld action (1) takes the schematic form

$$S \sim \int {\rm Tr}\,\sqrt{-\det\left(G + B + 2\pi\alpha' F\right)}\,. \tag{7}$$

In order to properly define this nonabelian analog of the Born-Infeld action (NBI), it is necessary to resolve the ordering ambiguities in (7). Since the spacetime coordinates X^a associated with the D-brane positions in space-time become themselves matrix-valued, even evaluating the pullbacks $G_{\alpha\beta}, B_{\alpha\beta}$ involves resolving ordering issues. Much work has been done recently to resolve these ordering ambiguities [46] but there is still no known definition of the nonabelian Born-Infeld theory (7) which is valid to all orders.

The nonabelian Born-Infeld action (7) becomes much simpler when, once again, the background space-time is assumed to be flat and we take the low-energy limit , leading to the nonabelian $U(N)$ super Yang-Mills action in $p+1$ dimensions. This action is the reduction to $p+1$ dimensions of the ten-dimensional $U(N)$ super Yang-Mills action (analogous to (5)). In this reduction, for $\alpha, \beta \leq p$, $F_{\alpha\beta}$ is the world-volume $U(N)$ field strength, and for $a > p, \alpha \leq p$, $F_{\alpha a} \rightarrow \partial_\alpha X^a$, where now A_α, X^a, and $F_{\alpha\beta}$ are $N \times N$ matrices. Since the derivatives ∂_a are set to zero in the dimensional reduction, we furthermore have, for $a, b > p$, $F_{ab} \rightarrow -i[X^a, X^b]$.

The low-energy description of a system of N coincident flat D-branes is thus given by $U(N)$ super Yang-Mills theory in the appropriate dimension. This connection between D-brane actions in string theory and super Yang-Mills theory has led to many new developments, including new insights into supersymmetric field theories, the M(atrix) theory and AdS/CFT correspondences, and brane world scenarios.

2.3 Branes from branes

In this subsection we describe a remarkable feature of D-brane systems: one or more D-branes of a fixed dimension can be used to construct additional D-branes of higher or lower dimension.

In our discussion of the D-brane action (1), we mentioned a group of terms $S_{\rm CS}$ which we did not describe explicitly. For a single Dp-brane, these Chern-Simons terms can be combined into a single expression of the form

$$S_{\rm CS} \sim \int_{\Sigma_{p+1}} \mathcal{A}\, e^{F+B}\,, \tag{8}$$

where $\mathcal{A} = \sum_k A^{(k)}$ represents a formal sum over all the Ramond-Ramond fields $A^{(k)}$ of various dimensions. In this integral, for each term $A^{(k)}$, the nonvanishing contribution to (8) is given by expanding the exponential of $F + B$ to order $(p+1-k)/2$, where the dimension of the resulting form saturates the dimension of the brane. For example, on a Dp-brane, there is a coupling of the form

$$\int_{\Sigma_{(p+1)}} A^{(p-1)} \wedge F\,. \tag{9}$$

This coupling implies that the $U(1)$ field strength on the Dp-brane couples to the RR field associated with $(p-2)$-branes. Thus, we can associate magnetic fields on a Dp-brane with dissolved $(p-2)$-branes living on the Dp-brane. This result generalizes to a system of multiple Dp-branes, in which case a trace is included on the right-hand side of (8). For example, on N compact Dp-branes wrapped on a p-torus, the flux

$$\frac{1}{2\pi} \int {\rm Tr}\; F_{\alpha\beta}, \tag{10}$$

of the magnetic field over a two-cycle on the torus is quantized and measures the number of units of D$(p-2)$-brane charge on the Dp-branes that threads the cycle integrated over. Thus, these branes are encoded in the field strength $F_{\alpha\beta}$. The object in (10) is the relevant component of the first Chern class of the $U(N)$ bundle described by the gauge field on the N branes. Similarly,

$$\frac{1}{8\pi^2} \int {\rm Tr}\; F \wedge F \tag{11}$$

encodes D$(p-4)$-brane charge on the Dp-branes, etc..

Just as lower-dimensional branes can be described in terms of the degrees of freedom associated with a system of N Dp-branes through the field strength $F_{\alpha\beta}$, higher-dimensional branes can be described by a system of N Dp-branes in terms of the commutators of the matrix-valued scalar fields X^a. Just as $\frac{1}{2\pi}F$ measures $(p-2)$-brane charge, the matrix

$$2\pi i[X^a, X^b] \tag{12}$$

measures $(p+2)$-brane charge [41, 47, 48]. The charge (12) should be interpreted as a form of local charge density. Just as the N positions of the Dp-branes are replaced by matrices in the nonabelian theory, so the locations of the charges become matrix-valued. The trace of (12) vanishes for finite sized matrices because the net Dp-brane charge of a finite-size brane configuration in flat spacetime vanishes. Higher multipole moments of the brane charge, however, have a natural definition in terms of traces of the charge matrix multiplied by powers of the scalars X^a, and generically are nonvanishing.

A simple example of the mechanism by which a system of multiple Dp-branes form a higher-dimensional brane is given by the matrix sphere. If we take a system of D0-branes with scalar matrices X^a given by

$$X^a = \frac{2r}{N} J^a, \qquad a = 1, 2, 3 \tag{13}$$

where J^a are the generators of $SU(2)$ in the N-dimensional representation, then we have a configuration corresponding to the "matrix sphere". This is a D2-brane of spherical geometry living on the locus of points satisfying $x^2 + y^2 + z^2 = r^2$. The "local" D2-brane charge of this brane is given by (12); here, for example, the D2-brane charge in the x-y plane is proportional to the matrix X^3 (z), as one would expect from the geometry of a spherical brane. The D2-brane configuration given by (13) is rotationally invariant (up to a gauge transformation). The restriction of the brane to the desired locus of points can be seen from the relation $(X^1)^2 + (X^2)^2 + (X^3)^2 = r^2 \mathbb{1} + \mathcal{O}(N^{-2})$.

2.4 T-duality

We conclude our discussion of D-branes with a brief description of T-duality. T-duality is a perturbative and nonperturbative symmetry which relates the type IIA and type IIB string theories. This duality symmetry was in fact crucial in the original discovery of D-branes [43]. A more detailed discussion of T-duality can be found in the textbook by Polchinski [49]. Using T-duality, we construct an explicit example of a brane within a brane encoded in super Yang-Mills theory, illustrating the ideas of the previous subsection. This example will be used in the following section to construct an analogous configuration with a tachyon.

Consider type IIA string theory on a spacetime of the form $M^9 \times S^1$ where M^9 is a generic 9-manifold of Lorentz signature, and S^1 is a circle of radius R. T-duality is the statement that this theory is precisely equivalent, even at the perturbative level, to type IIB string theory on the spacetime $M^9 \times (S^1)'$, where $(S^1)'$ is a circle of radius $R' = \alpha'/R$.

T-duality symmetry is most easily understood in the case of closed strings, where it amounts to an exchange of winding and momentum modes of the string. The string winding modes on S^1 have energy $R|m|/\alpha'$, where m is the winding number. The T-dual momentum modes on $(S^1)'$ have energy $|n|/R'$, where n is the momentum quantum number. These two sets of values coincide when m and n run over all possible integers. It is in fact straightforward to check that the full spectrum of closed string states is unchanged under T-duality. For the case of open strings, T-duality maps an open string with Neumann boundary conditions on S^1 to an open string with Dirichlet boundary conditions on $(S^1)'$, and vice versa. Thus, a Dp-brane which is wrapped around the circle S^1 is mapped under T-duality to a D$(p-1)$-brane which is localized to a point on the circle $(S^1)'$. Under T-duality the low-energy $(p+1)$-dimensional Yang-Mills theory on the p-brane is replaced by a p-dimensional Yang-Mills theory on the dual $(p-1)$-brane. Mathematically, the covariant derivative operator in the direction S^1 is replaced under T-duality with an adjoint scalar field X^a. Formally, this adjoint scalar field is an infinite size matrix [50], which contains information about the open strings wrapped an arbitrary number of times around the compact direction $(S^1)'$.

We can summarize the relevant mappings under T-duality in the following table

$$
\begin{array}{rcr}
\mathrm{IIA}/S^1 & \leftrightarrow & \mathrm{IIB}/(S^1)' \\
R & \leftrightarrow & R' = \alpha'/R \\
\text{Dirichlet/Neumann b.c.'s} & \leftrightarrow & \text{Neumann/Dirichlet b.c.'s} \\
p\text{-brane} & \leftrightarrow & (p \pm 1)\text{-brane} \\
2\pi\alpha'(i\partial_a + A_a) & \leftrightarrow & X^a
\end{array}
$$

The phenomena by which field strengths in one brane describe lower- or higher-dimensional branes can be easily understood using T-duality. The following simple example may help to clarify this connection. (For a more detailed discussion using this point of view, see Taylor [41].) constant magnetic

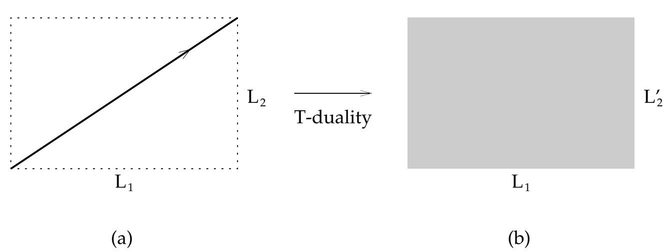


Figure 1: T-duality takes a diagonal D1-brane on a two-torus (a) to a D2-brane on the dual torus with constant magnetic flux encoding an embedded D0-brane (b).

Consider a D1-brane wrapped diagonally on a two-torus T^2 with sides of length $L_1 = L$ and $L_2 = 2\pi R$. (Figure 1(a)). This configuration is described in terms of the world-volume Yang-Mills theory on a D1-brane stretched in the L_1 direction through a transverse scalar field

$$X^2 = 2\pi R \xi_1 / L \,. \tag{14}$$

To be technically precise, this scalar field should be treated as an $\infty \times \infty$ matrix [50] whose (n, m) entry is associated with strings that connect the nth and mth images of the D1-brane on the covering space of S^1. The diagonal elements $X^2_{n,n}$ of this infinite matrix are given by $2\pi R(\xi_1 + nL)/L$, while all off-diagonal elements vanish. While the resulting matrix-valued function of ξ_1 is not periodic, it is periodic up to a gauge transformation

$$X^2(L) = V X^2(0) V^{-1} \tag{15}$$

where V is the shift matrix with nonzero elements $V_{n,n+1} = 1$.

Under T-duality in the x^2 direction the infinite matrix X^2_{nm} becomes the Fourier mode representation of a gauge field on a dual D2-brane:

$$A_2 = \frac{1}{R'L}\xi_1 \,. \tag{16}$$

The magnetic flux associated with this gauge field is

$$F_{12} = \frac{1}{R'L} \,, \tag{17}$$

so that

$$\frac{1}{2\pi}\int F_{12}\, d\xi^1\, d\xi^2 = 1 \,. \tag{18}$$

Note that the boundary condition (15) on the infinite matrix X^2 transforms under T-duality to the boundary condition on the gauge field

$$\begin{aligned} A_2(L, x_2) &= e^{2\pi i \xi_2/L'_2}\left(A_2(0, x_2) + i\partial_2\right) e^{-2\pi i \xi_2/L'_2} \\ &= A_2(0, x_2) + \frac{2\pi}{L'_2}, \end{aligned} \tag{19}$$

which (16) clearly satisfies. The off-diagonal elements of the shift matrix V in (15) describe winding modes which correspond after T-duality to the first Fourier mode $e^{2\pi i \xi_2/L'_2}$. The boundary condition on the gauge fields in the ξ_2 direction is trivial, which simplifies the T-duality map; a similar construction can be done with a nontrivial boundary condition in both directions, although the configuration looks more complicated in the D1-brane picture.

This construction gives a simple Yang-Mills description of the mapping of D-brane charges under T-duality: the initial configuration described above has charges associated with a single D1-brane wrapped around each of the directions of the 2-torus: D1_1+ D1_2. Under T-duality, these D1-branes are mapped to a D2-brane and a D0-brane respectively: D2_{12}+ D0. The flux integral (18) is the representation in the D2-brane world-volume Yang-Mills theory of the charge associated with a D0-brane which has been uniformly distributed over the surface of the D2-brane, just as in (10).

3 Tachyons and D-branes

We now turn to the subject of tachyons. Certain D-brane configurations are unstable, both in supersymmetric and nonsupersymmetric string theories. This instability is manifested as a tachyon, that is, as a state with $M^2 < 0$ in the spectrum of open strings that end on the D-brane. We will explicitly describe the tachyonic mode in the case of the open bosonic string in Section 4.1; this open bosonic string tachyon will be the focal point of most of the developments described in these notes. In this section we list some elementary D-brane configurations where tachyons arise, and we describe a particular situation in which the tachyon can be seen in the low-energy Yang-Mills description of the D-branes. This Yang-Mills background with a tachyon provides a simple field-theory model of a system analogous to the more complicated string field theory tachyon we describe in the later part of these notes. This simpler model may be useful to keep in mind in the later analysis.

3.1 D-brane configurations with tachyonic instabilities

Some simple examples of unstable D-brane configurations where the open string contains a tachyon include the following:

Brane-antibrane: A pair of parallel Dp-branes with opposite orientation in type IIA or IIB string theory which are separated by a distance $d \ll l_s$ give rise to a tachyon in the spectrum of open strings stretched between the branes [51]. The difference in orientation of the branes means that the two branes are really a brane and antibrane, carrying equal but opposite RR charges. Since the net RR charge is 0, the brane and antibrane can annihilate, leaving an uncharged vacuum configuration.

Wrong-dimension branes: In type IIA/IIB string theory, a Dp-brane of even/odd spatial dimension p is a stable BPS state with nonzero RR charge. On the other hand, a Dp-brane of the *wrong* dimension (*i.e.,* odd/even for IIA/IIB) carries no charges under the classical IIA/IIB supergravity fields, and has a tachyon in the open string spectrum. Such a brane can decay into the vacuum without violating charge conservation.

Bosonic D-branes: Like the wrong-dimension branes of IIA/IIB string theory, a Dp-brane of any dimension in the bosonic string theory carries no conserved charge and has a tachyon in the open string spectrum. Again, such a brane can decay into the vacuum without violating charge conservation.

3.2 Example: tachyon in low-energy field theory of two D-branes

In order to illustrate the physical behavior of tachyonic configurations, we consider in this subsection a simple example [52, 53] where a brane-antibrane tachyon can be seen in the context of the low-energy Yang-Mills theory.

The system we want to consider is a simple generalization of the (D2 + D0)-brane configuration we described using Yang-Mills theory in section 2.4. Consider a pair of D2-branes wrapped on a two-torus, one of which has a D0-brane embedded in it as a constant positive magnetic flux, and the other of which has an anti-D0-brane within it described by a constant negative magnetic flux. We take the two dimensions of the torus to be L_1, L_2. Following the discussion of Section 2.4, this configuration is equivalent under T-duality in the L_2 direction to a pair of crossed D1-branes (see Figure 2).

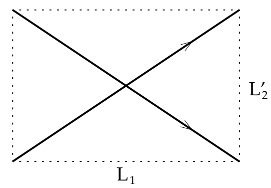


Figure 2: A pair of crossed D1-branes, T-dual to a pair of D2-branes with uniformly embedded D0- and anti-D0-branes.

The Born-Infeld energy of this configuration is

$$\begin{aligned} E_{\rm BI} &= 2\sqrt{(\tau_2 L_1 L_2)^2 + \tau_0^2} \\ &= \frac{1}{g}\left[\frac{2L_1L_2}{\sqrt{2\pi}} + \frac{(2\pi)^{3/2}}{L_1L_2} + \cdots\right], \end{aligned} \tag{20}$$

in units where $2\pi\alpha' = 1$. This can be computed either directly from the Born-Infeld action on the D2-branes (the abelian theory can be used since the matrices are diagonal), or by simply using the Pythagorean theorem in the T-dual D1-brane picture. The second term in the last line corresponds to the Yang-Mills approximation. In this approximation (dropping the D2-brane energy) the energy is

$$E_{\rm YM} = \frac{\tau_2}{4}\int {\rm Tr}\; F_{\alpha\beta}F^{\alpha\beta} = \frac{1}{4\sqrt{2\pi}g}\int {\rm Tr}\; F_{\alpha\beta}F^{\alpha\beta}\,. \tag{21}$$

We are interested in studying this configuration in the Yang-Mills approximation, in which we have a $U(2)$ theory on T^2 with field strength

$$F_{12} = \begin{pmatrix} \frac{2\pi}{L_1L_2} & 0 \\ 0 & -\frac{2\pi}{L_1L_2} \end{pmatrix} = \frac{2\pi}{L_1L_2}\tau_3\,. \tag{22}$$

This field strength can be realized as the curvature of a linear gauge field

$$A_1 = 0, \qquad A_2 = \frac{2\pi}{L_1L_2}\xi\tau_3\,, \tag{23}$$

which satisfies the boundary conditions

$$A_j(L,\xi_2) = \Omega(i\partial_j + A_j(0,\xi_2))\Omega^{-1}\,, \tag{24}$$

where

$$\Omega = e^{2\pi i(\xi_1/L_2)\tau_3}\,. \tag{25}$$

It is easy to check that this configuration indeed satisfies

$$E_{\rm YM} = \frac{1}{2g}\frac{(2\pi)^{3/2}}{L_1 L_2}{\rm Tr}\ \tau_3^2 = \frac{1}{g}\frac{(2\pi)^{3/2}}{L_1 L_2}\,, \tag{26}$$

as desired from (20). Since

$${\rm Tr}\ F_{\alpha\beta} = 0, \tag{27}$$

the gauge field we are considering is in the same topological equivalence class as $F = 0$. This corresponds to the fact that the D0-brane and anti-D0-brane can annihilate. To understand the appearance of the tachyon, we can consider the spectrum of excitations δA_α around the background (23) [52]. The eigenvectors of the quadratic mass terms in this background are described by torus theta functions which satisfy boundary conditions related to (24). There are precisely two elements in the spectrum with the negative eigenvalue $-4\pi/L_1L_2$. These theta functions [52] are tachyonic modes of the theory which are associated with the annihilation of the positive and negative fluxes that encode the D0- and anti-D0-brane. These tachyonic modes are perhaps easiest to understand in the dual configuration, where they provide a direction of instability in which the two crossed D1-branes reconnect as in Figure 3.

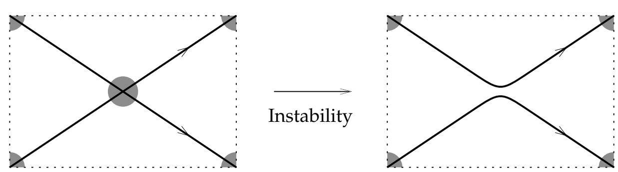


Figure 3: The brane-antibrane instability of a D0-$\bar{\rm D}0$ system embedded in two D2-branes, as seen in the T-dual D1-brane picture.

It is also interesting to note that in the T-dual picture the tachyonic modes of the gauge field have support which is localized near the two intersection points and take the off-diagonal form

$$\delta A_t \sim \begin{pmatrix} 0 & \star \\ \star & 0 \end{pmatrix}, \tag{28}$$

which naturally encodes our geometric understanding that the tachyonic modes reconnect the two D1-branes near each intersection point.

The full Yang-Mills action around the background (23) can be written as a quartic function of the mass eigenstates around this background. Written in terms of these modes, there are nontrivial cubic and quartic terms which couple the tachyonic modes to all the massive modes in the system. If we integrate out the massive modes, we know from the topological reasoning above that an effective potential arises for the tachyonic mode A_t, with a maximum value of (26) and a minimum value of 0. This system is highly analogous to the bosonic open string tachyon we will discuss in the remainder of these lectures. Our current understanding of the bosonic string through bosonic string field theory is analogous to that of someone who only knows the Yang-Mills theory around the background (23) in terms of a complicated quartic action for an infinite family of modes. Without knowledge of the topological structure of the theory, and given only a list of the coefficients in the quartic action, such an individual would have to systematically calculate the tachyon effective potential by explicitly integrating out all the massive modes one by one. This would give a numerical approximation to the minimum of the effective potential, which could be made arbitrarily good by raising the mass of the cutoff at which the effective action is computed. It may be helpful to keep this example in mind in the following sections, where an analogous tachyonic system is considered in string field theory. For further discussion of this unstable configuration in Yang-Mills theory, see the references [52, 53, 54, 55].

3.3 The Sen conjectures

The existence of the tachyonic mode in the open bosonic string indicates that the standard choice of perturbative vacuum for this theory is unstable. In the early days of the subject, there were some calculations suggesting that this tachyon could condense, leading to a more stable vacuum [56]. Kostelecky and Samuel argued early on that the stable vacuum could be identified in string field theory in a systematic way [57], however there was no clear physical picture for the significance of this stable vacuum. In 1999, Ashoke Sen reconsidered the problem of tachyons in string field theory. Sen suggested that the open bosonic string should really be thought of as living on a D25-brane, and hence that the perturbative vacuum for this string theory should have a nonzero vacuum energy associated with the tension of this D25-brane. He suggested that the tachyon is simply the instability mode of the D25-brane, which carries no conserved charge and hence is not expected to be stable, as discussed in section 3.1. More precisely, Sen conjectured that the following three statements are true [39]:

1. The tachyon potential has a locally stable minimum, whose energy density $\mathcal{E}$, measured with respect to that of the unstable critical point, is equal to minus the tension of the D25-brane:
$$\mathcal{E} = -T_{25}\,. \tag{29}$$

2. Lower-dimensional D-branes are solitonic solutions of the string theory on the background of a D25-brane.

3. The locally stable vacuum of the system is the closed string vacuum. In this vacuum the D25-brane is absent and no conventional open string excitations exist.

It was also suggested by Sen that open string field theory was a natural setup to test the above conjectures. He sharpened the first conjecture by suggesting that Witten's OSFT should precisely reproduce the tension of the D25-brane, which he expressed in terms of the open string coupling constant g which appears in the formulation of open string field theory:

$$T_{25} = \frac{1}{2\pi^2 g^2}\,. \tag{30}$$

We will give the instructive derivation of this result in section 7.

Our first encounter with the tachyon conjectures will happen in section 5, where we calculate the first nontrivial term in the tachyon potential, find a minimum, and discover that, even with this rough approximation, the calculated $\mathcal{E}$ gives about 70% of the expected answer. In Section 7 of these lectures we systematically explore the evidence for these conjectures in Witten's OSFT. First, however, we need to develop the technical tools to do specific calculations in string field theory.

4 Witten's cubic open string field theory

In this and in the following two sections we give a detailed description of Witten's open string field theory [26]. This section contains a general introduction to this string field theory. Subsection 4.1 reviews the quantization of the open bosonic string in 26 dimensions and sets notation. Subsection 4.2 gives a heuristic introduction to open string field theory, which follows Witten's original paper. In subsection 4.3 we discuss the algebraic structure of open string field theory which emerges naturally in the context of conformal field theory. This discussion also develops the properties of the twist operator Ω which reverses the orientation of open strings.

The work in the present section prepares the ground for sections 5 and 6, in which precise definitions of the open bosonic SFT are given using conformal field theory and the mode decomposition of overlap equations. For further background on Witten's OSFT see the reviews of LeClair, Peskin and Preitschopf [58], of Thorn [59], and of Gaberdiel and Zwiebach [60].

4.1 The bosonic open string

In this subsection we review the quantization of the open bosonic string. For further details see the textbooks by Green, Schwarz, and Witten [61] and by Polchinski [49]. The bosonic open string can be quantized using the BRST approach starting from the action

$$S = -\frac{1}{4\pi\alpha'}\int \sqrt{-\gamma}\gamma^{ab}\partial_a X^\mu \partial_b X_\mu, \tag{31}$$

where γ is an auxiliary dynamical metric on the world-sheet. This action can be gauge-fixed to conformal gauge $\gamma_{ab} \sim \delta_{ab}$, introducing at the same time ghost and antighost fields $c^\pm(\sigma), b_{\pm\pm}(\sigma)$. The gauge-fixed action is

$$S = -\frac{1}{4\pi\alpha'}\int \partial_a X^\mu \partial^a X_\mu + \frac{1}{\pi}\int \left(b_{++}\partial_- c^+ + b_{--}\partial_+ c^-\right)\,. \tag{32}$$

The matter fields X^μ can be expanded in modes using

$$X^\mu(\sigma,\tau) = x_0^\mu + 2p^\mu\tau + i\sqrt{2}\sum_{n\neq 0}\frac{1}{n}\alpha_n^\mu \cos(n\sigma)e^{-in\tau}\,, \tag{33}$$

where we have fixed $l_s = \sqrt{2\alpha'} = \sqrt{2}$, so that $\alpha' = 1$. In the quantum theory, x_0^μ and p^μ obey the canonical commutation relations

$$[x_0^\mu, p^\nu] = i\eta^{\mu\nu}\,. \tag{34}$$

The α_n^μ's with negative/positive values of n become raising/lowering operators for the oscillator modes on the string. They satisfy the Hermiticity conditions $(\alpha_n^\mu)^\dagger = \alpha_{-n}^\mu$ and the commutation relations

$$[\alpha_m^\mu, \alpha_n^\nu] = m\eta^{\mu\nu}\delta_{m+n,0}\,. \tag{35}$$

We will often use the canonically normalized oscillators:

$$a_n^\mu = \frac{1}{\sqrt{n}}\alpha_n^\mu\,, \quad n \geq 1\,, \tag{36}$$

which obey the commutation relations

$$[a_m^\mu, a_n^{\nu\dagger}] = \eta^{\mu\nu}\delta_{m,n}\,. \tag{37}$$

We will also frequently use position modes x_n for $n \neq 0$ and lowering and raising operators $a_0, a_0^\dagger$ for the zero modes. These are related to the modes in (33) through (dropping space-time indices)

$$\begin{aligned} x_n &= \frac{i}{\sqrt{2n}}(a_n - a_n^\dagger) \\ x_0 &= \frac{i}{2}(a_0 - a_0^\dagger) \end{aligned} \tag{38}$$

The ghost and antighost fields can be decomposed into modes through

$$\begin{aligned} c^\pm(\sigma) &= \sum_n c_n e^{\mp in\sigma} \\ b_{\pm\pm}(\sigma) &= \sum_n b_n e^{\mp in\sigma}\,. \end{aligned} \tag{39}$$

The ghost and antighost modes satisfy the anticommutation relations

$$\begin{aligned} \{c_n, b_m\} &= \delta_{n+m,0} \\ \{c_n, c_m\} &= \{b_n, b_m\} = 0\,. \end{aligned} \tag{40}$$

A general state in the open string Fock space can be written in the form

$$\alpha_{-n_1}^{\mu_1}\cdots\alpha_{-n_i}^{\mu_i}\; c_{-m_1}\cdots c_{-m_j}\; b_{-p_1}\cdots b_{-p_l}\;|0;k\rangle \tag{41}$$

where

$$n_i \geq 1\,, \quad m_i \geq -1\,, \quad \text{and} \quad p_i \geq 2\,, \tag{42}$$

since $|0;k\rangle$ is annihilated by

$$\begin{aligned} b_n|0;k\rangle &= 0, \quad n \geq -1 \\ c_n|0;k\rangle &= 0, \quad n \geq 2\,, \\ \alpha_n^\mu|0;k\rangle &= 0, \quad n \geq 1\,. \end{aligned} \tag{43}$$

The state $|0;k\rangle$ is a momentum eigenstate:

$$p^\mu|0;k\rangle = k^\mu|0;k\rangle\,. \tag{44}$$

The zero-momentum state $|0;0\rangle$ is the SL(2,R) invariant vacuum; we will often write it simply as $|0\rangle$. This vacuum is defined to have ghost number 0, and it is normalized by the equation

$$\langle 0;k|c_{-1}c_0c_1|0;k'\rangle = (2\pi)^{26}\delta(k-k') \tag{45}$$

For string field theory we will also find it convenient to work with the vacua of ghost number one and two:

$$\begin{aligned} G=1: &\qquad |0_1\rangle = c_1|0\rangle \\ G=2: &\qquad |0_2\rangle = c_0c_1|0\rangle\,. \end{aligned} \tag{46}$$

In the notation of Polchinski [49], these two vacua are written as

$$\begin{aligned} |0_1\rangle &= |0\rangle_m \otimes |\downarrow\rangle \\ |0_2\rangle &= |0\rangle_m \otimes |\uparrow\rangle\,, \end{aligned} \tag{47}$$

where $|0\rangle_m$ is the matter vacuum and $|\downarrow\rangle, |\uparrow\rangle$ are the ghost vacua annihilated by b_0, c_0.

The BRST operator of this theory is given by

$$Q_B = \sum_{n=-\infty}^{\infty} c_n L_{-n}^{(\mathrm{m})} + \sum_{n,m=-\infty}^{\infty} \frac{(m-n)}{2} : c_m c_n b_{-m-n} : - c_0 \tag{48}$$

where the matter Virasoro operators are given by

$$L_q^{(\mathrm{m})} = \begin{cases} \frac{1}{2}\sum_n \alpha_{q-n}^\mu \alpha_{\mu\,n}, & q \neq 0 \\ p^2 + \sum_{n=1}^\infty \alpha_{-n}^\mu \alpha_{\mu\,n}\,. \end{cases} \tag{49}$$

4.2 Witten's cubic bosonic SFT

The discussion of the previous subsection leads to a systematic quantization of the open bosonic string in the conformal field theory framework. Using this approach it is possible, in principle, to calculate an arbitrary perturbative on-shell scattering amplitude for physical string states. To study tachyon condensation in string theory, however, we require a nonperturbative, off-shell formalism for the theory— a string field theory.

A very simple form for the off-shell open bosonic string field theory action was proposed by Witten in 1986: [26]

$$S = -\frac{1}{2}\int \Psi \star Q\Psi - \frac{g}{3}\int \Psi \star \Psi \star \Psi \,. \tag{50}$$

This action has the general form of a Chern-Simons theory on a 3-manifold, although for string field theory there is no explicit interpretation of the integration in terms of a concrete 3-manifold. In Eq. (50), g is interpreted as the (open) string coupling constant. The field Ψ is a string field, which takes values in a graded algebra $\mathcal{A}$. Associated with the algebra $\mathcal{A}$ there is a star product

$$\star : \mathcal{A} \otimes \mathcal{A} \to \mathcal{A}, \tag{51}$$

under which the degree G is additive ($G_{\Psi\star\Phi} = G_\Psi + G_\Phi$). There is also a BRST operator

$$Q : \mathcal{A} \to \mathcal{A}, \tag{52}$$

of degree one ($G_{Q\Psi} = 1 + G_\Psi$). String fields can be integrated using

$$\int : \mathcal{A} \to \mathbf{C}\,. \tag{53}$$

This integral vanishes for all Ψ with degree $G_\Psi \neq 3$. Thus, the action (50) is only nonvanishing for a string field Ψ of degree 1.

The elements $Q, \star, \int$ that define the string field theory are assumed to satisfy the following axioms:

(a) Nilpotency of Q: $Q^2\Psi = 0, \quad \forall \Psi \in \mathcal{A}$.

(b) $\int Q\Psi = 0, \quad \forall \Psi \in \mathcal{A}$.

(c) Derivation property of Q:
$Q(\Psi \star \Phi) = (Q\Psi) \star \Phi + (-1)^{G_\Psi} \Psi \star (Q\Phi), \quad \forall \Psi, \Phi \in \mathcal{A}$.

(d) Cyclicity: $\int \Psi \star \Phi = (-1)^{G_\Psi G_\Phi} \int \Phi \star \Psi, \quad \forall \Psi, \Phi \in \mathcal{A}$.

(e) Associativity: $(\Phi \star \Psi) \star \Xi = \Phi \star (\Psi \star \Xi), \quad \forall \Phi, \Psi, \Xi \in \mathcal{A}$.

When these axioms are satisfied, the action (50) is invariant under the gauge transformations

$$\delta\Psi = Q\Lambda + \Psi \star \Lambda - \Lambda \star \Psi\,, \tag{54}$$

for any gauge parameter $\Lambda \in \mathcal{A}$ with degree 0.

When the string coupling g is taken to vanish, the equation of motion for the theory defined by (50) simply becomes $Q\Psi = 0$, and the gauge transformations (54) simply become

$$\delta\Psi = Q\Lambda\,. \tag{55}$$

Thus, when $g = 0$ this string field theory gives precisely the structure needed to describe the free bosonic string. The motivation for introducing the extra structure in (50) was to find a simple interacting extension of the free theory, consistent with the perturbative expansion of open bosonic string theory.

Witten presented this formal structure and argued that all the needed axioms are satisfied when $\mathcal{A}$ is taken to be the space of string fields

$$\mathcal{A} = \{\Psi[x(\sigma); c(\sigma), b(\sigma)]\} \tag{56}$$

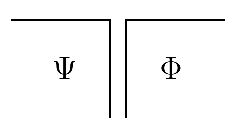
which can be described as functionals of the matter, ghost and antighost fields describing an open string in 26 dimensions with $0 \leq \sigma \leq \pi$. Such a string field can be written as a formal sum over open string Fock space states with coefficients given by an infinite family of space-time fields

$$\Psi = \int d^{26}p \ \left[\phi(p)\,|0_1;p\rangle + A_\mu(p)\ \alpha^\mu_{-1}|0_1;p\rangle + \cdots\right] \tag{57}$$

Each Fock space state is associated with a given string functional, just as the states of a harmonic oscillator are associated with wavefunctions of a particle in one dimension. For example, the matter ground state $|0\rangle_m$ annihilated by a_n for all $n \geq 1$ is associated (up to a constant C) with the functional of matter modes

$$|0\rangle_m \rightarrow C \exp\left(-\frac{1}{2}\sum_{n>0}^{\infty} n x_n^2\right) . \tag{58}$$

For Witten's cubic string field theory, the BRST operator Q in (50) is the usual open string BRST operator Q_B, given in (48), and the degree associated with a Fock space state is the ghost number of that state. The star product $\star$ acts on a pair of functionals Ψ, Φ by gluing the right half of one string to the left half of the other using a delta function interaction



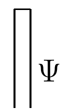
This star product factorizes into separate matter and ghost parts. In the matter sector, the star product is given by the formal functional integral

$$\begin{aligned}
&(\Psi \star \Phi)\,[z(\sigma)] \\
&\quad \equiv \int \prod_{0 \leq \tilde{\tau} \leq \frac{\pi}{2}} dy(\tilde{\tau})\ dx(\pi - \tilde{\tau}) \prod_{\frac{\pi}{2} \leq \tau \leq \pi} \delta[x(\tau) - y(\pi - \tau)]\ \Psi[x(\tau)]\Phi[y(\tau)]\,, \\
&\qquad\qquad x(\tau) = z(\tau) \quad \text{for} \quad 0 \leq \tau \leq \frac{\pi}{2}\,, \\
&\qquad\qquad y(\tau) = z(\tau) \quad \text{for} \quad \frac{\pi}{2} \leq \tau \leq \pi\,.
\end{aligned} \tag{59}$$

Similarly, the integral over a string field factorizes into matter and ghost parts, and in the matter sector is given by

$$\int \Psi = \int \prod_{0 \le \sigma \le \pi} dx(\sigma) \prod_{0 \le \tau \le \frac{\pi}{2}} \delta[x(\tau) - x(\pi - \tau)] \; \Psi[x(\tau)] \,. \tag{60}$$

This corresponds to gluing the left and right halves of the string together with a delta function interaction



The ghost sector of the theory is defined in a similar fashion, but has an anomaly due to the curvature of the Riemann surface that describes the three-string vertex. The ghost sector can be described either in terms of fermionic ghost fields $c(\sigma), b(\sigma)$ or through bosonization in terms of a single bosonic scalar field $\phi_g(\sigma)$. From the functional point of view of Eqs. (59, 60), it is easiest to describe the ghost sector in the bosonized language. In this language, the ghost fields $b(\sigma)$ and $c(\sigma)$ are replaced by the scalar field $\phi_g(\sigma)$, and the star product in the ghost sector is given by (59) with an extra insertion of $\exp(3i\phi_g(\pi/2)/2)$ inside the integral. Similarly, the integration of a string field in the ghost sector is given by (60) with an insertion of $\exp(-3i\phi_g(\pi/2)/2)$ inside the integral. Witten first described the cubic string field theory using bosonized ghosts. While this approach is useful for some purposes, we will use fermionic ghost fields in the remainder of these lecture notes. With the fermionic ghosts, there is no need for insertions at the string midpoint.

The expressions (59, 60) may seem rather formal, as they are written in terms of functional integrals. These expressions, however, can be given precise meaning when described in terms of creation and annihilation operators acting on the string Fock space. In Sections 5 and 6 we give a more precise definition of the string field theory action using conformal field theory and the countable mode decomposition of the string.

4.3 Algebraic structure of OSFT

Here we discuss an approach to the algebraic structure of OSFT that is inspired by conformal field theory. This approach can be used to write rather general string actions, including those whose interactions are not based on delta-function overlaps. In this language the string action takes the form

$$S(\Phi) = -\frac{1}{g^2}\left[\frac{1}{2}\langle\, \Phi\,,\, Q\,\Phi\rangle + \frac{1}{3}\langle\, \Phi\,,\, \Phi * \Phi\rangle\right]. \tag{61}$$

Here g is the open string coupling constant, the string field Φ is a state in the total matter plus ghost CFT, Q is the kinetic operator, $*$ denotes a multiplication or star-product, and $\langle\cdot\,,\cdot\rangle$ is a bilinear inner product on the state space of the CFT. We will discuss the relationship between the action (61) and the form of the action (50) used in the previous subsection shortly.

The kinetic operator Q satisfies the following identities

$$\begin{aligned}
&Q^2 A = 0, \\
&Q(A * B) = (QA) * B + (-1)^A A * (QB)\,, \\
&\langle\, QA, B\,\rangle = -(-)^A \langle A, QB\rangle\,.
\end{aligned} \tag{62}$$

The first equation is the nilpotency condition, the second states that Q is a derivation of the star product, and the third states that Q is self-adjoint. There are also identities involving the inner product and the star operation

$$\begin{aligned}
&\langle A, B\rangle = (-)^{AB}\langle B, A\rangle\,, \\
&\langle\, A\,, B * C\,\rangle = \langle A * B\,,\, C\,\rangle \\
&A * (B * C) = (A * B) * C\,.
\end{aligned} \tag{63}$$

In the sign factors, the exponents $A, B, \cdots$ denote the Grassmanality of the state, and should be read as $(-)^A \equiv (-)^{\epsilon(A)}$ where $\epsilon(A) = 0\,(\text{mod } 2)$ when A is Grassmann even, and $\epsilon(A) = 1\,(\text{mod } 2)$ when A Grassmann odd. The first property above is a symmetry condition, the second indicates that the inner product has a cyclicity property analogous to the similar property of the trace operation. Finally, the last equation is the statement that the star product is associative.

Finally, we also have that the star operation is an *even* product of degree zero (as before, we identify degree with ghost number). In plain english, this means that both the grassmanality and the ghost number of the star product of two string fields is obtained from those of the string fields without any additional offset:

$$\begin{aligned}
&\epsilon(A * B) = \epsilon(A) + \epsilon(B)\,, \\
&\text{gh}(A * B) = \text{gh}(A) + \text{gh}(B)\,.
\end{aligned} \tag{64}$$

In this language Q is an odd operator of degree one:

$$\begin{aligned}
&\epsilon(QA) = \epsilon(A) + 1\,, \\
&\text{gh}(QA) = \text{gh}(A) + 1\,.
\end{aligned} \tag{65}$$

In the conventions we shall work the SL(2,R) vacuum $|0\rangle$ is assigned ghost number zero. The Grassmanality $\epsilon(A)$ of a string field A is an integer mod 2. In open string field theory, Grassmanality and ghost number (degree) are related because Grassmann odd operators carry odd units of ghost number.

The algebraic structure discussed here is very similar, but not identical to that in section 4.2. The string field Φ and the action (61) can be related to the string field Ψ and the action (50) of the previous section by taking

$$\Phi = g\Psi \tag{66}$$

and by relating the inner product used here to the integral used in (50) through

$$\langle A\,, B\rangle = \int A \star B\,. \tag{67}$$

The first two conditions in (62) are then clearly equivalent to properties (a) and (c) of section 4.2. You can also readily see that the first two properties in (63) hold given properties (d) and (e). Property (b), however, does not have a counterpart in this formalism. A counterpart exists if we assume the existence of a suitable identity string field $\mathcal{I}$, as we will discuss at the end of this subsection.

Throughout these lectures we will go back and forth between the CFT notation with string field Φ and action (61) and the oscillator description with string field Ψ and action (50) (which we rewrite more explicitly as (148) in section 6). While we could have chosen to use one notation and neglect the other, both formalisms are used extensively in the literature, and some results are more easily expressed in one notation than the other. When in doubt, the reader should return to the previous paragraph to see how the two notations are related.

Let us now deduce some basic properties of the string field, in particular its ghost number and its Grassmanality. The Grassmanality of Φ can be deduced from the condition that the kinetic term of the string action must be non-vanishing. Using the above properties we have

$$\langle \Phi, Q\Phi\rangle = (-1)^{\Phi(1+\Phi)}\langle Q\Phi, \Phi\rangle = \langle Q\Phi, \Phi\rangle = -(-1)^{\Phi}\langle \Phi, Q\Phi\rangle\,. \tag{68}$$

It is clear that the string field Φ must be Grassmann odd. At this point we must use some CFT knowledge to decide on the Grassmanality of the SL(2,R) vacuum and on the ghost number of the string field. For bosonic strings we have that zero momentum tachyon states are of the form $tc_1|0\rangle$, where c_1 is a ghost field oscillator. Since this oscillator is Grassmann odd, and the string field is also Grassmann odd, we must declare the SL(2,R) vacuum to be Grassmann even. Thus

$$|0\rangle \text{ is a Grassmann even state of ghost number zero}\,. \tag{69}$$

Since the c_1 oscillator carries ghost number one, we also deduce that the open string field must have ghost number one.

$$|\Phi\rangle \text{ is a Grassmann odd state of ghost number one}\,. \tag{70}$$

Equations (62), (63), (64), (65), and (70) guarantee that the string field action is invariant under the gauge transformations:

$$\delta\Phi = Q\Lambda + \Phi * \Lambda - \Lambda * \Phi\,, \tag{71}$$

for any Grassmann-even ghost-number zero state Λ. Moreover, variation of the action gives the field equation

$$Q\Phi + \Phi * \Phi = 0\,. \tag{72}$$

Exercise Verify that the string action in (61) is gauge invariant under the transformations (71).

It is convenient to use the above structures to define a multilinear object that given three string fields yields a number:

$$\langle\, A\,,\, B\,,\, C\rangle \equiv \langle\, A\,,\, B * C\,\rangle \tag{73}$$

The middle equation in (63) implies the *cyclicity* of the multilinear form. A small calculation immediately gives:

$$\langle\, A\,,\, B\,,\, C\rangle = (-)^{A(B+C)}\langle\, B\,,\, C\,,\, A\rangle \tag{74}$$

A basic consistency check of the signs above is that the cubic term $\langle\, \Phi\,,\, \Phi\,,\, \Phi\rangle$ in the action (62) is strictly cyclic for odd Φ, and therefore does not vanish.

Open string theory has additional algebraic structure that sometimes plays a crucial role. One such structure arises from the twist operation, which reverses the parametrization of a string. From the algebraic viewpoint this is summarized by the existence of an operator Ω that satisfies the following properties:

$$\begin{aligned}
&\Omega(QA) = Q(\Omega A)\\
&\langle\, \Omega A\,,\, \Omega B\,\rangle = \langle\, A\,,\, B\,\rangle\\
&\Omega\,(A * B) = (-)^{AB+1}\,\Omega(B) * \Omega(A)\,.
\end{aligned} \tag{75}$$

The first property means that the BRST operator has zero twist, or does not change the twist property of the states it acts on. The second property states that the bilinear form is twist invariant. The third property is crucial. Up to signs, twisting the star product of string fields amounts to multiplying the twisted states in *opposite order*. This change of order is a simple consequence of the basic multiplication rule where the second half of the first string must be glued to the first half of the second one. The sign factor is also important. For the string field Φ, which is grassmann odd, it gives

$$\Omega(\Phi * \Phi) = +\,(\Omega\Phi) * (\Omega\Phi) \tag{76}$$

with the plus sign. This result, together with the first two equations in (75) immediately implies that the string field action in (62) is twist invariant:

$$S(\Omega\Phi) = S(\Phi)\,. \tag{77}$$

This invariance under twist transformations allows one to construct new string theories by truncating the spectrum to the subset of states that are twist even. Moreover, in solving the string field equations it will be possible to find consistent solutions by restricting oneself to the twist even subspace of the string field.

Exercise. Letting Ω_A denote the Ω eigenvalue of A, show that

$$\langle A, B, C\rangle = \Omega_A\Omega_B\Omega_C(-1)^{AB+BC+CA+1}\langle C, B, A\rangle\,. \tag{78}$$

Exercise. Let $\Omega A_{\pm} = \pm A$ and $\epsilon(A_{\pm}) = 1$. Show that

$$\langle A_+, A_+, A_- \rangle = 0\,. \tag{79}$$

Exercise. We will see later that the star product of the vacuum with itself is the vacuum plus Virasoro descendents:

$$|0\rangle * |0\rangle = |0\rangle + \cdots \tag{80}$$

Show that this implies that the vacuum is twist odd:

$$\Omega|0\rangle = -|0\rangle. \tag{81}$$

The star algebra may have an identity element $\mathcal{I}$. If $\mathcal{I}$ exists, it is presumed to satisfy

$$\mathcal{I} * A = A * \mathcal{I} = A\,, \tag{82}$$

for all states A. Some properties of $\mathcal{I}$ are immediately deduced from the above definition:

$$\mathcal{I} \text{ is Grassmann even, ghost number zero, twist odd string field.} \tag{83}$$

The twist odd property follows from the twist property of products

$$\Omega(\mathcal{I} * A) = (-1)^{0\cdot A+1}(\Omega A) * (\Omega\mathcal{I}) = -(\Omega A) * (\Omega\mathcal{I})\,. \tag{84}$$

Since the left hand side is also just (ΩA) it must follow that

$$\Omega\mathcal{I} = -\mathcal{I}\,. \tag{85}$$

This is consistent with the fact that the SL(2,R) vacuum is also twist odd. Indeed the identity string field is just the vacuum plus Virasoro descendents of the vacuum, as we shall see in Section 8.4.

Any derivation D of the star algebra should annihilate the identity:

$$D(\mathcal{I} * A) = (D\mathcal{I}) * A + \mathcal{I} * DA = (D\mathcal{I}) * A + DA\,. \tag{86}$$

Since the left hand side also equals DA, one concludes that $(D\mathcal{I}) * A = 0$ for all A, and thus the expectation that $D\mathcal{I} = 0$. In the star algebra of open strings an identity state has been identified [62, 63, 64] that satisfies the expected properties for most well-behaved states.

Finally, if the identity string field is annihilated by the derivation Q, then

$$\langle Q\Psi\,, \mathcal{I}\rangle = -(-)^{\Psi}\langle \Psi, Q\mathcal{I}\rangle = 0\,. \tag{87}$$

The identification (67) then yields

$$0 = \int Q\Psi \star \mathcal{I} = \int Q\Psi\,, \tag{88}$$

which is property (b) in the axiomatic formulation of OSFT discussed in section 4.2.

5 String field theory: conformal field theory approach

A direct conformal field theory evaluation of the string action is perhaps the most economical way to proceed in the case of simple computations. We will explain this definition of the action, using at the same time the example of the action restricted to only the tachyon field to illustrate the definitions. The string action, written before in (61) is given by

$$S(\Phi) = -\frac{1}{g^2}\left[\frac{1}{2}\langle\,\Phi\,,\,Q\,\Phi\rangle + \frac{1}{3}\langle\,\Phi\,,\,\Phi * \Phi\rangle\right]. \tag{89}$$

This OSFT action can be used to describe the spacetime field theory of any D-brane. For example, for a Dp-brane we would have an underlying conformal field theory of a field X^0 and p fields X^i with Neumann boundary conditions, and $(25-p)$ fields X^a with Dirichlet boundary conditions. In our computations, we will assume that the brane has unit volume, in which case the mass M of the brane coincides with its tension. One can show that, in units where $\alpha' = 1$,

$$M = \frac{1}{2\pi^2}\frac{1}{g^2}\,. \tag{90}$$

We will prove this result in section 7.1.

We will evaluate the OSFT action by truncating the string field down to the zero momentum tachyon. The systematic approximation of the full theory by successive level truncation is described in detail in Section 7.3. In the level expansion this zero momentum tachyon is assigned level zero. The tachyon vertex operator is $e^{ipX(z)}c(z)$ and the associated state is $c_1|0;p\rangle$. The zero momentum tachyon state is $c_1|0;0\rangle$ or in simpler notation $c_1|0\rangle$. Since we have

$$L_0 c_1|0\rangle = -c_1|0\rangle\,, \tag{91}$$

the level ℓ of a state is related to the L_0 eigenvalue as

$$\ell = L_0 + 1\,. \tag{92}$$

The string field truncated to the zero momentum tachyon is written as

$$|T\rangle = t\,c_1|0\rangle\,, \tag{93}$$

where the variable t denotes the expectation value of the tachyon field, and it is a spacetime constant. The variable t is related to the tachyon field ϕ in the expansion (57) through

$$t = g\phi(0)\,. \tag{94}$$

As we alternate between notation Ψ and Φ for the string field, we will use ϕ and t for the zero-momentum tachyon. After truncating to just the tachyon degree of freedom t the tachyon potential $V(t)$ is just minus $S(|T\rangle)$ and thus

$$V(t) = -S(|T\rangle) = M(2\pi^2)\Big(\frac{1}{2}\langle T, QT\rangle + \frac{1}{3}\langle T, T, T\rangle\Big)\,. \tag{95}$$

In fact, it is convenient to define the ratio

$$f(t) \equiv \frac{V(t)}{M} = (2\pi^2)\Big(\frac{1}{2}\langle T, QT\rangle + \frac{1}{3}\langle T, T, T\rangle\Big)\,. \tag{96}$$

The function $f(t)$ is a rescaled version of the tachyon potential. By construction, $f(t)$ has a quadratic term and a cubic term, so $f(t=0) = 0$. The Sen conjecture requires that $f(t)$ have a critical point at $t = t^*$ that satisfies

$$f(t^*) = -1\,. \tag{97}$$

This is indeed equivalent to saying that the energy difference between the D-brane vacuum and the stable vacuum equals the energy M of the D-brane. It suggests strongly that the stable vacuum is a vacuum without a D-brane. It is perhaps useful to remark that $V(t)$ as obtained directly from the OSFT action does not convey the true gravitational picture where absolute vacuum energies are important. The vacuum with the D-brane, namely at $t = 0$ has a positive cosmological constant, or vacuum energy. This is in fact the D-brane energy. As the theory rolls to the stable vacuum, the vacuum energy goes to zero. Thus the tachyon potential $V(t)$ is missing an additive constant, which becomes important when coupling to gravity (which we will not consider in the present lectures). Such a constant term at least morally belongs in a more general OSFT action where the disk partition function would naturally appear as a field independent contribution to the string action. This disk partition function calculated with the boundary condition appropriate to the D-brane is in fact proportional to the D-brane energy.

5.1 Kinetic term computations

Let us begin the computation of the string action truncated to the tachyon by evaluating $\langle T, QT\rangle$. To this end we need to use the normalization condition

$$\langle 0|c_{-1}c_0c_1|0\rangle = 1\,, \tag{98}$$

which is appropriate if we compactify all coordinates (including time) into circles of unit circumference. Indeed, comparing with (45), we see that the right-hand side of (98) should have a $(2\pi)^{26}\delta(0)$, which is equivalent to the full spacetime volume V. In our full compactification, $V = 1$. The compactification of time is only a formal trick that facilitates computations but is not strictly necessary.

Exercise: Given $c(z) = \sum_n \frac{c_n}{z^{n-1}}$ show that

$$\langle 0|c(z_1)c(z_2)c(z_3)|0\rangle = (z_1 - z_2)(z_1 - z_3)(z_2 - z_3)\,. \tag{99}$$

Now that we must compute precisely we should make clear the CFT definition of the inner product

Definition: $\langle A, B\rangle = \langle bpz(A)|B\rangle$. Here $bpz : \mathcal{H} \to \mathcal{H}^*$ is BPZ conjugation, which we review next.

Given a primary field $\phi(z)$ of dimension d, it has a mode expansion

$$\phi(z) = \sum_n \frac{\phi_n}{z^{n+d}} \quad \rightarrow \quad \phi_n = \oint \frac{dz}{2\pi i} z^{n+d-1}\phi(z)\,. \tag{100}$$

We define

$$bpz(\phi_n) \equiv \oint \frac{dt}{2\pi i} t^{n+d-1}\phi(t)\,, \qquad \text{with} \quad t = -\frac{1}{z}\,. \tag{101}$$

Note that this simply defines the BPZ conjugation of the oscillator with the same formula as the oscillator itself (100) but referred to a coordinate at $z = \infty$. This integral is evaluated by using the transformation law

$$\phi(t)(dt)^d = \phi(z)(dz)^d\,. \tag{102}$$

We therefore get

$$bpz(\phi_n) \equiv -\oint \frac{dz}{2\pi i}\frac{1}{z^2}\Big(-\frac{1}{z}\Big)^{n+d-1}\phi(z)(z^2)^d\,. \tag{103}$$

The minus sign in front arises from a reversal of contour of integration (a contour circling $t = 0$ clockwise circles $z = 0$ counterclockwise). Moreover the transformation law was used to reexpress $\phi(t)$ in terms of the field $\phi(z)$ whose mode expansion is given. Simplifying the integral one finds

$$bpz(\phi_n) = (-1)^{n+d}\oint \frac{dz}{2\pi i} z^{-n+d-1}\phi(z) \;= (-1)^{n+d}\phi_{-n}\,. \tag{104}$$

In summary, we have shown that

$$bpz(\phi_n) = (-1)^{n+d}\phi_{-n}\,. \tag{105}$$

This equation defines BPZ conjugation when we supplement it with the rule

$$bpz\Big(\phi_{n_1}\cdots\phi_{n_p}|0\rangle\Big) = \langle 0|bpz(\phi_{n_1})\cdots bpz(\phi_{n_p})\,. \tag{106}$$

This formula is correct as stated also when the oscillators are anticommuting. The only condition for its validity is that the various modes with mode numbers of the same sign must commute (or anticommute). Otherwise BPZ conjugation produces a sequence of oscillators in *reverse* order.

A nontrivial example of the above rules arises when we calculate the BPZ conjugates of the modes L_n of the stress tensor. Although the stress tensor $T(z)$ is not a primary field, it transforms as a primary under SL(2,C) transformations, and therefore it does transform as a dimension two primary under the inversion needed in the definition of BPZ. Thus we have

$$bpz(L_n) = (-1)^n L_{-n}\,, \tag{107}$$

and for a string of oscillators we must write

$$bpz\Big(L_{n_1}\cdots L_{n_p}|0\rangle\Big) = \langle 0|bpz(L_{n_p})\cdots bpz(L_{n_1})\,. \tag{108}$$

Since $c(z)$ has dimension minus one, $bpz(c_1) = (-1)^{1+1}c_{-1} = c_{-1}$, so $bpz(c_1|0\rangle) = \langle 0|c_{-1}$. With this we have

$$\langle T, QT \rangle = t^2 \langle 0|c_{-1}Qc_1|0\rangle \,. \tag{109}$$

Because of the form of the inner product only the term c_0L_0 in Q can contribute and we have

$$\langle T, QT \rangle = t^2 \langle 0|c_{-1}c_0L_0c_1|0\rangle = -t^2 \langle 0|c_{-1}c_0c_1|0\rangle = -t^2 \,. \tag{110}$$

This completes the computation of the quadratic term in the tachyon potential. The negative sign obtained is the expected one, showing the instability of the $t = 0$ field configuration.

5.2 Interaction term computation

To compute the interaction of three tachyons we must explain how the three vertex is defined in CFT language. Consider three states A, B, and C and their associated vertex operators $\mathcal{O}_A$, $\mathcal{O}_B$, and $\mathcal{O}_C$. We define

$$\langle A, B, C \rangle \equiv \Big\langle f_1^D \circ \mathcal{O}_A(0), f_2^D \circ \mathcal{O}_B(0),\, f_3^D \circ \mathcal{O}_C(0) \Big\rangle_D \tag{111}$$

Here the right hand side denotes the CFT correlator of the conformal transforms of the vertex operators $\mathcal{O}_A$, $\mathcal{O}_B$, and $\mathcal{O}_C$. The conformal transforms are specified by the functions f_i as we explain now. Let there be three canonical coordinates ξ_i, with $i = 1, 2, 3$. The three functions $f_i(\xi_i)$ define maps from the upper half disks $\Im(\xi_i) \geq 0, |\xi_i| \leq 1$ into a disk D, with the points $\xi_i = 0$ being taken into points in the boundary of the disk D. The meaning of the conformal map of operators is that: $f_i \circ \mathcal{O}_A(0)$ *is the operator* $\mathcal{O}_A(\xi_i = 0)$ *expressed in terms of local operators at* $f_i(\xi_i = 0)$. The disk D may have the form of a unit disk, or can be the (conformally equivalent) upper half plane, or any other arbitrary form. Of course, the unit disk and the upper half plane are especially convenient for explicit computations.

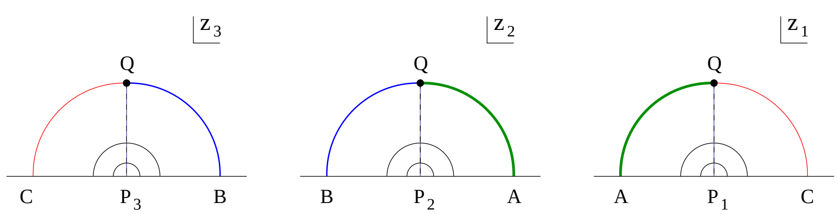


Figure 4: Representation of the cubic vertex as the gluing of 3 half–disks.

For the SFT at hand, the picture is given in Fig.4. The worldsheets of the three strings are represented as the unit half–disks $\{|\xi_i| \leq 1, \Im\,\xi \geq 0\}$, $i = 1, 2, 3$, in three copies of the complex plane. The boundaries $|\xi_i| = 1$ in the respective upper half-disks are *the* strings. Thus the point $\xi_i = i$ is the string midpoint. The interaction defining the vertex is built by gluing the three half-disks to form a single disk. This is done by the half-string identifications:

$$
\begin{aligned}
\xi_1\xi_2 &= -1\,, \qquad \text{for } |\xi_1| = 1, \quad \Re(\xi_1) \le 0\,,\\
\xi_2\xi_3 &= -1\,, \qquad \text{for } |\xi_2| = 1, \quad \Re(\xi_2) \le 0\,,\\
\xi_3\xi_1 &= -1\,, \qquad \text{for } |\xi_3| = 1, \quad \Re(\xi_3) \le 0\,.
\end{aligned} \tag{112}
$$

Note that the common interaction point Q, is indeed $\xi_i = i$ (for $i = 1, 2, 3$), namely the mid–point of each open string $|\xi_i| = 1$, $\Im(\xi_i) \ge 0$. The left-half of the first string is glued with the right-half of the second string, and the same is repeated cyclically. This construction defines a specific 'three–punctured disk', a genus zero Riemann surface with a boundary, three marked points (punctures) on this boundary, and a choice of local coordinates ξ_i around each puncture.

The calculation of the functions $f_i^D(\xi)$ require a choice of disk D. We begin with the case when the disk D is simply chosen to be the interior of the unit disk $|w| < 1$, as shown in Fig. 5. In this case the functions $f_i^{D_w} \equiv f_i$ must map each half-disk to a 120° wedge of this unit disk. To construct the explicit maps that send ξ_i to the w plane, one notices that the SL(2,C) transformation

$$
h(z) = \frac{1 + i\xi}{1 - i\xi}\,, \tag{113}
$$

maps the unit upper–half disk $\{|\xi| \le 1, \Im\xi \ge 0\}$ to the 'right' half–disk $\{|h| \le 1, \Re h \ge 0\}$, with $z = 0$ going to $h(0) = 1$. Thus the functions

$$
\begin{aligned}
f_1(\xi_1) &= e^{\frac{2\pi i}{3}} \left(\frac{1 + i\xi_1}{1 - i\xi_1}\right)^{\frac{2}{3}}\,,\\
f_2(\xi_2) &= \left(\frac{1 + i\xi_2}{1 - i\xi_2}\right)^{\frac{2}{3}}\,,\\
f(\xi_3) &= e^{-\frac{2\pi i}{3}} \left(\frac{1 + i\xi_3}{1 - i\xi_3}\right)^{\frac{2}{3}}\,,
\end{aligned} \tag{114}
$$

will send the three half-disks to three wedges in the w plane of Fig. 5, with punctures at $e^{\frac{2\pi i}{3}}$, 1, and $e^{-\frac{2\pi i}{3}}$ respectively. This specification of the functions $f_i(\xi_i)$ gives the definition of the cubic vertex. In this representation cyclicity (*i.e.*, $\langle \Phi_1, \Phi_2, \Phi_3 \rangle = \langle \Phi_2, \Phi_3, \Phi_1 \rangle$) is manifest by construction. By SL(2,C) invariance, there are many other possible representations that give exactly the same off–shell amplitudes.

A useful choice is obtained by mapping the interacting w disk symmetrically to the upper half z-plane H. This is the convention that we shall mostly be using. We can therefore define the functions f_i^H by composition of the earlier maps f_i (that send the half-disks to the w unit disk) with the map $h^{-1}(w) = -i\,\frac{w-1}{w+1}$ that takes this unit disk to the upper–half–plane, with the three punctures on the real axis (Fig. 6),

$$
f_1^H(\xi_1) \;\equiv\; h^{-1} \circ f_1(\xi_1) = S(f_3^H(\xi_1))
$$

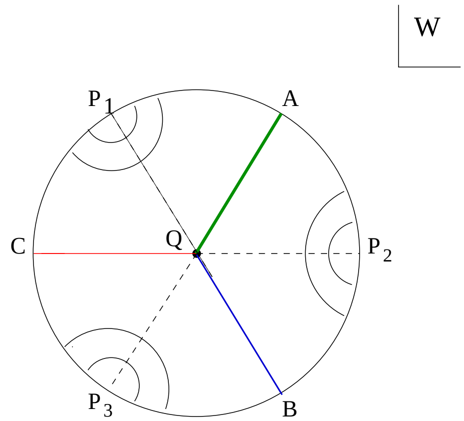


Figure 5: Representation of the cubic vertex as a 3–punctured unit disk.

$$
\begin{aligned}
&= \sqrt{3} + \frac{8}{3}\,\xi_1 + \frac{16}{9}\,\sqrt{3}\,\xi_1{}^2 + \frac{248}{81}\,\xi_1{}^3 + O\left(\xi_1{}^4\right)\,. \\
f_2^H(\xi_2) &\equiv h^{-1}\circ f_2(\xi_2) = S(f_1^H(\xi_2)) = \tan\left(\frac{2}{3}\,\arctan(\xi_2)\right) \\
&= \frac{2}{3}\,\xi_2 - \frac{10}{81}\,\xi_2{}^3 + O\left(z_2{}^5\right)\,. \\
f_3^H(\xi_3) &\equiv h^{-1}\circ f_3\,(\xi_3) = S(f_2^H(\xi_3)) \\
&= -\sqrt{3} + \frac{8}{3}\,\xi_3 - \frac{16}{9}\,\sqrt{3}\,\xi_3{}^2 + \frac{248}{81}\,\xi_3{}^3 + O(\xi_3{}^4)\,. \qquad (115)
\end{aligned}
$$

The three punctures are at $f_1^H(0) = +\sqrt{3}$, $f_2^H(0) = 0$, $f_3^H(0) = -\sqrt{3}$, and the SL(2,R) map $S(z) = \frac{z-\sqrt{3}}{1+\sqrt{3}z}$ cycles them (thus $S\circ S\circ S(z) = z$).

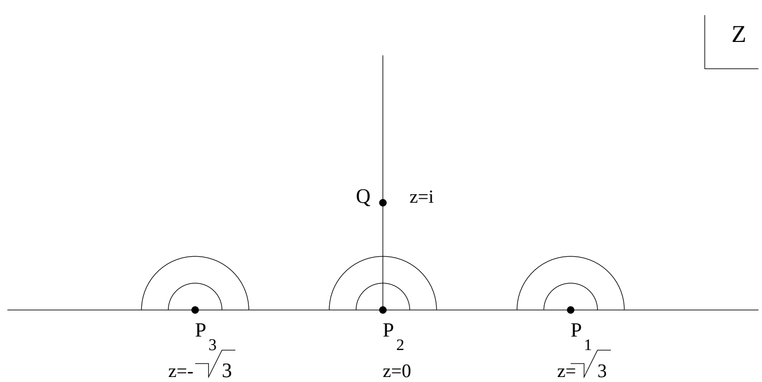


Figure 6: Representation of the cubic vertex as the upper–half plane with 3 punctures on the real axis.

This completes the definition of the string field theory action. When the disk D is presented as a unit disk the functions f_i in (111) are the functions given in equation (114).

When the disk D is presented as the upper half plane H the relevant functions in (111) are the functions f_i^H given in (115) above.

Exercise: Verify explicitly by a *by-hand* calculation that the first two terms in the expansion of f_1^H and f_3^H, as well as the first term in f_2^H are correct.

Let us now return to the computation of the tachyon action. For our string field $|T\rangle = tc_1|0\rangle$ the interaction term $\langle T, T, T\rangle$ will be given by

$$\langle T, T, T\rangle = t^3 \langle c_1, c_1, c_1\rangle \,. \tag{116}$$

Since the vertex operator associated to $c_1|0\rangle$ is $c(z)$, using (111) we write:

$$\langle T, T, T\rangle = t^3 \langle f_1^H \circ c(0), f_2^H \circ c(0), f_3^H \circ c(0)\rangle_H \tag{117}$$

Since the field $c(z)$ is a primary of dimension minus one, we have

$$\frac{c(z)}{dz} = \frac{c(\xi)}{d\xi} \quad \to c(\xi) = \frac{c(z)}{\frac{dz}{d\xi}} \tag{118}$$

Therefore

$$f \circ c(0) \equiv c(\xi = 0) = \frac{c(f(0))}{f'(0)}. \tag{119}$$

Using equations (115) to read the values of $f_1^H(0)$ and $\frac{df_1^H}{d\xi}(0)$ we therefore get, for example,

$$f_1^H \circ c(0) = \frac{c(f_1^H(0))}{f_1^{H\prime}(0)} = \frac{c(\sqrt{3})}{8/3} \,. \tag{120}$$

The other two insertions are dealt with similarly, and we find

$$\begin{aligned} \langle T, T, T\rangle &= t^3 \Big\langle \frac{c(\sqrt{3})}{8/3}, \frac{c(0)}{2/3}, \frac{c(-\sqrt{3})}{8/3} \Big\rangle_H \\ &= \frac{3^3}{2^7} \langle c(\sqrt{3}) c(0) c(-\sqrt{3})\rangle_H = \frac{3^4\sqrt{3}}{2^6} \,, \end{aligned} \tag{121}$$

where in the last step we made use of (99). Our answer is therefore

$$\langle T, T, T\rangle = \frac{81\sqrt{3}}{64} t^3 \equiv t^3 K^3 \,, \quad \langle c_1, c_1, c_1\rangle = \frac{81\sqrt{3}}{64} = K^3 \,. \tag{122}$$

This completes the calculation of an interaction term.

5.3 A first test of the tachyon conjecture

Having obtained the kinetic term of the tachyon truncated action in (110) and the cubic term in (122) we are now in a position to evaluate the function $f(t)$ in (96):

$$f(t) = 2\pi^2 \Big(-\frac{1}{2} t^2 + \frac{1}{3} K^3 t^3 \Big) \,. \tag{123}$$

We must now find the (locally) stable critical point $t = t^*$ of this potential and evaluate the value of $f(t^*)$. It is clear the answer will not be the precise one $f = -1$, since we have truncated the string field dramatically. Nevertheless, we hope to get an answer that is reasonably close, if level expansion is supposed to make sense.

The equation of motion is

$$-t^* + t^{*2}K^3 = 0 \quad \to \quad t^* = \frac{1}{K^3}\,, \tag{124}$$

and substituting back we find

$$f(t^*) = -\frac{1}{3}\frac{\pi^2}{K^6} = -\pi^2\frac{2^{12}}{3^{10}} = -\pi^2\frac{4096}{59049} \simeq -0.684. \tag{125}$$

Thus is this simplest approximation, where we only kept the tachyon zero mode we have found that the critical point cancels about 70% of the D-brane energy. In section (7.3) we discuss the extension of this calculation to include massive string modes.

5.4 String vertex in the CFT approach: Neumannology

When doing explicit computations in OSFT we need to consider interactions of fields other than the tachyon. The explicit computation of the previous section becomes a lot more involved for massive fields, and it is useful to find an automated procedure to deal with such calculations. One such procedure is based on conformal field theory conservation laws. This is a very effective method, but we will not review it here since its explanation in Rastelli and Zwiebach [65] is self-contained. Another approach uses the explicit Fock representations of the string vertex. This will be our subject of interest here. We will provide a self-contained derivation of the Neumann coefficients that define the three string vertex both in the matter and in the ghost sector. In fact, our construction will be general and applies to three string interactions other than the one used in OSFT. We will determine the full structure of the three string vertex, except for the matter zero modes.

In the Fock space representation of the vertex, we must find a state $\langle V_3| \in \mathcal{H}^* \otimes \mathcal{H}^* \otimes \mathcal{H}^*$ such that for any Fock space states A, B and C one finds that

$$\langle A, B, C\rangle \equiv \langle V_3|A\rangle_{(1)}|B\rangle_{(2)}|B\rangle_{(3)}\,. \tag{126}$$

Since we provided in (111) a definition of the left hand side of the above equation, the vertex $\langle V_3|$ is implicitly defined. Our procedure will be general in that the functions $f_r(\xi)$ that map the canonical half-disks to the upper half plane will be kept arbitrary. There is a natural ansatz for the vertex:

$$\begin{aligned}\langle V_3| \;=\; & \mathcal{N}(\langle 0|c_{-1}c_0)^{(3)})(\langle 0|c_{-1}c_0)^{(2)})(\langle 0|c_{-1}c_0)^{(1)}) \\ & \exp\Big(-\frac{1}{2}\sum_{r,s}\sum_{n,m\geq 1}\alpha_m^{(r)}\,N_{mn}^{rs}\,\alpha_n^{(s)}\Big)\exp\Big(\sum_{r,s}\sum_{\substack{m\geq 0\\ n\geq 1}} b_m^{(r)}\,X_{mn}^{rs}\,c_n^{(s)}\Big)\,.\end{aligned} \tag{127}$$

Here $\mathcal{N}$ is a normalization factor, which will be determined shortly. In fact, its determination is essentially the tachyon computation of the previous section. Moreover, note that the nontrivial oscillator dependence in the matter sector is in the form of an exponential of a quadratic form. This is a general result that follows from the free field property of the matter CFT. Having just a quadratic form is possible also for the ghost sector, but it requires a careful choice of vacua. This is because there is a sum rule regarding ghost number– if the vacua are not chosen conveniently, extra linear ghost factors are necessary in the vertex. Since the vertex state $\langle V_3|$ is a bra we use out-vacua, in particular the vacua $\langle 0|c_{-1}c_0$. This is quite convenient because the ghost number conservation law is satisfied when each of the states A, B and C in (126) is of ghost number one. Indeed in each of the three state spaces we must have a total ghost number of three– two are supplied by the out-vacuum, and one by the in-state. This clearly allows the nontrivial ghost dependence of the vertex to be just a pure exponential with zero ghost number. A final point concerns the sum restrictions over the ghost oscillators. These simply arise because only oscillators that do not kill the vacua $\langle 0|c_{-1}c_0$ should appear in the exponential. Thus for the antighost oscillators b_m we find $m \geq 0$ and for the ghost oscillators c_n we find $n \geq 1$

The normalization factor $\mathcal{N}$ can be determined by finding the overlap of the vertex with three zero momentum tachyons $c_1|0\rangle$. In this case we have

$$\langle c_1, c_1, c_1\rangle = \langle V_3|c_1\rangle_{(1)}|c_1\rangle_{(2)}|c_1\rangle_{(3)} = \mathcal{N}\,, \tag{128}$$

since all oscillators in the exponentials kill the zero momentum tachyon. In (122) we found the value of this constant for the case of the OSFT vertex

$$\mathcal{N} = K^3 = \frac{3^{9/2}}{2^6}\,. \tag{129}$$

The calculation in the general case is not any more complicated and it is a good exercise!

Exercise: Show that for arbitrary functions $f_i(\xi)$, $i = 1, 2, 3$, that map half-disks to the UHP, the constant $\mathcal{N}$ in the vertex (127) is given by:

$$\mathcal{N} = \frac{(f_1(0) - f_2(0))(f_1(0) - f_3(0))(f_2(0) - f_3(0))}{f_1'(0)f_1'(0)f_1'(0)}\,. \tag{130}$$

Our goal now is to find explicit expressions for the Neumann coefficients N^{rs}_{mn} and X^{rs}_{mn} in terms of the functions f_i that define the vertex.

We begin with the matter sector, where the following conventions are used

$$i\partial X(z) = \sum \frac{\alpha_n}{z^{n+1}}\,, \quad \alpha_n = \oint \frac{dz}{2\pi i}\, z^n\, i\partial X\,, \tag{131}$$

$$\langle i\partial X(z)\; i\partial X(w)\rangle = \frac{1}{(z-w)^2}\,, \quad [\alpha_n, \alpha_m] = n\delta_{m+n,0}\,. \tag{132}$$

To find the matter Neumann coefficients we evaluate

$$M = \langle V_3|\; R\left(i\partial X^{(r)}(z)\; i\partial X^{(s)}(w)\right)\; c_1^{(1)}|0\rangle_{(1)}\, c_1^{(2)}|0\rangle_{(2)}\, c_1^{(3)}|0\rangle_{(3)} \tag{133}$$

in two different ways. In here $R(\dots)$ denotes radial ordering, necessary when $r = s$. For our first computation we use the mode expansion (131) of the conformal fields to find that

$$M = \langle V_3| \Big(\sum_{m,n} \frac{1}{z^{-m+1}} \frac{1}{w^{-n+1}} \alpha^{(r)}_{-m} \alpha^{(s)}_{-n} + \frac{\delta^{rs}}{(z-w)^2} \Big) c_1^{(1)}|0\rangle_{(1)} \, c_1^{(2)}|0\rangle_{(2)} \, c_1^{(3)}|0\rangle_{(3)} \tag{134}$$

and the oscillator form (127) of the vertex to obtain

$$M = -\mathcal{N} \sum_{m,n} z^{m-1} w^{n-1} \, mn N^{rs}_{mn} + \mathcal{N} \frac{\delta^{rs}}{(z-w)^2} \, . \tag{135}$$

In the second evaluation we first rewrite M as

$$M = \langle V_3| \; i\partial X^{(r)}(z) \; i\partial X^{(s)}(w) \, c^{(1)}(0) c^{(2)}(0) c^{(3)}(0) |0\rangle_{(1)} \, |0\rangle_{(2)} \, |0\rangle_{(3)} \, , \tag{136}$$

and reinterpret as a correlator, in the spirit of (111):

$$M = \Big\langle f_r \circ i\partial X(z) \; f_s \circ i\partial X(w) \; f_1 \circ c(0) \; f_2 \circ c(0) \; f_3 \circ c(0) \Big\rangle \, . \tag{137}$$

The ghost part of this correlator gives the factor $\mathcal{N}$. The matter part, using $i\partial X(z) = i\partial X(f(z)) \frac{df}{dz}$, and (132) finally gives

$$M = \mathcal{N} \, f_r'(z) \, f_s'(w) \Big\langle \; i\partial X(f_r(z)) \; i\partial X(f_s(w)) \; \Big\rangle = \mathcal{N} \frac{f_r'(z) \, f_s'(w)}{(f_r(z) - f_s(w))^2} \, . \tag{138}$$

Equating the results (135) and (138) of the two evaluations of M we obtain:

$$\sum_{m,n} z^{m-1} w^{n-1} \, mn N^{rs}_{mn} - \frac{\delta^{rs}}{(z-w)^2} = - \frac{f_r'(z) \, f_s'(w)}{(f_r(z) - f_s(w))^2} \, . \tag{139}$$

It is now simple to pick up the coefficients N^{rs}_{mn} by contour integration over small circles surrounding $z = 0$ and $w = 0$. The second term on the left-hand side gives no contribution, and one finally finds

$$N^{rs}_{mn} = - \frac{1}{mn} \oint_0 \frac{dz}{2\pi i} \frac{1}{z^m} \oint_0 \frac{dw}{2\pi i} \frac{1}{w^n} \; \frac{f_r'(z) \, f_s'(w)}{(f_r(z) - f_s(w))^2} \, . \tag{140}$$

This is the desired expression for the Neumann coefficients of the matter sector. They can be used for an arbitrary vertex. The above contour integrals are straightforward to compute and they can be easily done by a computer in a series expansion. In terms of residues the expression above is equivalent to

$$N^{rs}_{mn} = - \frac{1}{mn} \text{Res}_{z=0} \text{Res}_{w=0} \Big[\frac{1}{z^m} \frac{1}{w^n} \; \frac{f_r'(z) \, f_s'(w)}{(f_r(z) - f_s(w))^2} \Big] \tag{141}$$

Exercise: Show that the contour integrals in (140) can be evaluated in any order. Do this both for the case when $r \neq s$ and for the case when $r = s$.

We now turn to the calculation of the ghost Neumann coefficients X^{rs}_{mn}. For this we need mode expansions and two point functions for the ghost CFT:

$$c(z) = \sum_n \frac{c_n}{z^{n-1}}\,, \quad b(z) = \sum_n \frac{b_n}{z^{n+2}}\,, \quad \langle c(z)b(w)\rangle = \frac{1}{z-w}\,. \tag{142}$$

The strategy is once more based on the computation of a certain expression in two different ways. Indeed, we consider the overlap

$$G = \langle V_3|\, R\left(b^{(s)}(z)\ c^{(r)}(w)\right)\, |\, c_1^{(1)}|0\rangle_{(1)}\, c_1^{(2)}|0\rangle_{(2)}\, c_1^{(3)}|0\rangle_{(3)} \tag{143}$$

and first evaluate it by using the mode expansion of the antighost and ghost fields, and then the explicit expression for the vertex in (127). In this way we find

$$\begin{aligned} G &= \langle V_3|\Big(\sum_{m,n} \frac{1}{z^{-n+2}}\frac{1}{w^{-m-1}}\, b^{(s)}_{-n}\, c^{(r)}_{-m} + \frac{w}{z(z-w)}\Big)\ c_1^{(1)}|0\rangle_{(1)}\, c_1^{(2)}|0\rangle_{(2)}\, c_1^{(3)}|0\rangle_{(3)} \\ &= \mathcal{N}\sum_{m,n} z^{-n+2}w^{-m-1}\, X^{rs}_{mn} + \mathcal{N}\frac{w}{z(z-w)}\,. \end{aligned} \tag{144}$$

In the second computation G is interpreted as a correlator and we have

$$\begin{aligned} G &= \Big\langle f_s\circ b(z)\ f_r\circ c(w)\ f_1\circ c(0)\ f_2\circ c(0)\ f_3\circ c(0)\Big\rangle \qquad (145)\\ &= \frac{(f_s'(z))^2}{f_r'(w)}\frac{1}{f_1'(0)f_2'(0)f_3'(0)}\Big\langle b(f_s(z))\ c(f_r(w))\ c(f_1(0))\, c(f_2(0))\, c(f_3(0))\Big\rangle\,, \end{aligned}$$

where we used the standard conformal maps of the relevant operators, all of which are primary. The final correlator is in the upper half plane and all the field arguments refer to the coordinates in the upper half plane. The correlator can be calculated by using OPE's, but it is simpler to use the singularity structure and derive the normalization from a special configuration. Note, for example, that there must be zeroes when any pair of c fields approach each other. In particular, this will include a factor $(f_1(0)-f_2(0))(f_1(0)-f_3(0))(f_2(0)-f_3(0))$ as in $\mathcal{N}$ (see (130)). We will also have poles when the antighost approaches any ghost. These considerations imply that

$$G = \mathcal{N}\frac{(f_s'(z))^2}{f_r'(w)}\frac{1}{f_s(z)-f_r(w)}\frac{\prod_{I=1}^3(f_r(w)-f_I(0))}{\prod_{J=1}^3(f_s(z)-f_J(0))}\,. \tag{146}$$

We can now equate the results obtained in (144) and (146). Picking up the coefficients via contour integration, and noting that the second term on the right-hand side of (144) does not contribute for the relevant values of m and n, we find

$$X^{rs}_{mn} = \oint\frac{dz}{2\pi i}\frac{1}{z^{n-1}}\oint\frac{dw}{2\pi i}\frac{1}{w^{m+2}}\,\frac{(f_s'(z))^2}{f_r'(w)}\frac{1}{f_s(z)-f_r(w)}\frac{\prod_{I=1}^3(f_r(w)-f_I(0))}{\prod_{J=1}^3(f_s(z)-f_J(0)}\,. \tag{147}$$

This is the general result for the ghost Neumann coefficients. Again, for any vertex they are easily calculated by power series expansions and picking up residues. For particular vertices one can simplify somewhat the above expressions and find interesting relations. In fact, a fair amount of work can be done for the OSFT vertex in simplifying the above results. One can show that the matrices N and X are related, and while no closed form expressions are known for the coefficients, they can be generated quite efficiently from the simpler expressions.

Since any specific Neumann coefficient can be calculated exactly with a finite number of operations, the exact computation of $\langle A, B, C\rangle$ for any three Fock space states A, B, and C requires a finite number of operations, as well.

6 SFT action: oscillator approach

In this section, we give a more detailed discussion of Witten's open bosonic string field theory from the oscillator point of view. The main goal of this section is to explicitly formulate the OSFT action in the string Fock space, where the action (50) takes the form

$$S = -\frac{1}{2}\langle V_2|\Psi, Q\Psi\rangle - \frac{g}{3}\langle V_3|\Psi,\Psi,\Psi\rangle\,. \qquad (148)$$

In this expression, $\langle V_2|$ and $\langle V_3|$ are elements of the two-fold and three-fold product of the dual Fock space $(\mathcal{H}^*)^2$ and $(\mathcal{H}^*)^3$, respectively. These objects defined in terms of the string Fock space give a rigorous definition to the abstract action (50) through the replacement

$$\begin{aligned} \langle V_2|A, B\rangle &\rightarrow \int A\star B \\ \langle V_3|A, B, C\rangle &\rightarrow \int A\star B\star C\,. \end{aligned}$$

Subsection 6.1 is a warmup, in which we review some basic features of the simple harmonic oscillator and discuss squeezed states. In subsection 6.2 we relate modes on the full string to modes on half strings, giving formulae needed to compute the three-string vertex. In subsection 6.3 we derive the two-string vertex in oscillator form, and in subsection 6.4 we give an explicit formula for the three-string vertex. In subsection 6.5 we put these pieces together and discuss the calculation of the full SFT action.

6.1 Squeezed states and the simple harmonic oscillator

Let us consider a simple harmonic oscillator with annihilation operator

$$a = -i\left(\sqrt{\frac{\alpha}{2}}x + \frac{1}{\sqrt{2\alpha}}\partial_x\right) \qquad (149)$$

where α is an arbitrary constant. The oscillator ground state is associated with the wave-function

$$|0\rangle \rightarrow \left(\frac{\alpha}{\pi}\right)^{1/4} e^{-\alpha x^2/2}\,. \qquad (150)$$

In the harmonic oscillator basis $|n\rangle$, the Dirac position basis states $|x\rangle$ have a squeezed state form

$$|x\rangle = \left(\frac{\alpha}{\pi}\right)^{1/4} \exp\left(-\frac{\alpha}{2}x^2 - i\sqrt{2\alpha}a^\dagger x + \frac{1}{2}(a^\dagger)^2\right)|0\rangle\,. \tag{151}$$

A general wavefunction is associated with a state through the correspondence

$$f(x) \to \int_{-\infty}^{\infty} dx\; f(x)|x\rangle\,. \tag{152}$$

In particular, we have

$$\begin{aligned} \delta(x) \quad &\to \quad \left(\frac{\alpha}{\pi}\right)^{1/4} \exp\left(\frac{1}{2}(a^\dagger)^2\right)|0\rangle\,, \\ 1 \quad &\to \quad \int dx\;|x\rangle = \left(\frac{4\pi}{\alpha}\right)^{1/4} \exp\left(-\frac{1}{2}(a^\dagger)^2\right)|0\rangle\,. \end{aligned} \tag{153}$$

This shows that the delta and constant functions both have squeezed state representations in terms of the harmonic oscillator basis. The norm of a squeezed state

$$|s\rangle = \exp\left(\frac{1}{2}s(a^\dagger)^2\right)|0\rangle \tag{154}$$

is given by

$$\langle s|s\rangle = \frac{1}{\sqrt{1-s^2}}\,. \tag{155}$$

The states (153) are non-normalizable, but since they have $s = \pm 1$, they are right on the border of normalizability. The states (153) can be used to calculate, just like we do with the Dirac basis states $|x\rangle$, which lie outside the single-particle Hilbert space.

It will be useful for us to generalize the foregoing considerations in several ways. A particularly simple generalization arises when we consider a pair of degrees of freedom x, y described by a two-harmonic oscillator Fock space basis. In such a basis, repeating the preceding analysis leads us to a function-state correspondence for the delta functions relating x, y of the form

$$\delta(x \pm y) \to \exp\left(\pm a^\dagger_{(x)} a^\dagger_{(y)}\right)(|0\rangle_x \otimes |0\rangle_y)\,. \tag{156}$$

Note that this result is independent of α; like the δ function, the resulting state is again non-normalizable. we will find these squeezed state expressions very useful in describing the two- and three-string vertices of Witten's open string field theory. It is worth pointing out here that there are several ways of deriving (156). The most straightforward way is to carry out a two-dimensional Gaussian integral analogous to (153). We can also derive (156) indirectly, however (at least up to an overall constant) from the following argument. From the general result that delta functions give squeezed states, we expect that up to an overall constant

$$\begin{aligned} &\delta(x \pm y) \to \\ &\quad |D_\pm\rangle = \exp\left(\pm\frac{1}{2}\left[A a^\dagger_{(x)} a^\dagger_{(x)} + 2B a^\dagger_{(x)} a^\dagger_{(y)} + C a^\dagger_{(y)} a^\dagger_{(y)}\right]\right)(|0\rangle_x \otimes |0\rangle_y)\,. \end{aligned} \tag{157}$$

The state associated with the delta function must satisfy the constraints

$$\begin{aligned} (x \pm y)|D_\pm\rangle &= 0 \\ (p_x \mp p_y)|D_\pm\rangle &= 0\,. \end{aligned} \tag{158}$$

Rewriting x, p_x in terms of $a_{(x)}, a^\dagger_{(x)}$ and similarly for y, p_y, these conditions impose the constraints

$$\begin{aligned} \left[(A \pm B - 1)a^\dagger_{(x)} + (B \pm C \pm 1)a^\dagger_{(y)}\right]|D_\pm\rangle &= 0 \\ \left[(A \mp B + 1)a^\dagger_{(x)} + (B \mp C \mp 1)a^\dagger_{(y)}\right]|D_\pm\rangle &= 0\,, \end{aligned} \tag{159}$$

from which it follows that $A = C = 0$ and $B = \pm 1$, reproducing (156) up to an overall constant. We will use this indirect method, following Gross and Jevicki, to derive the three-string vertex in subsection 6.4.

6.2 Half-string modes

For many computations it is useful to think of the string as being "split" into a left half and a right half. Formally, the string field can be expressed as a functional $\Psi[L, R]$, where L, R describe the left and right parts of the string. This is a very appealing idea, since it leads to a simple picture of the star product in terms of matrix multiplication

$$(\Psi \star \Phi)[L, R] = \int \mathcal{D}A\ \Psi[L, A]\Phi[A, R]\,. \tag{160}$$

While there has been quite a bit of work aimed at making this "split string" formalism precise [66, 67, 68], the technical details in this approach become quite complicated when one attempts to precisely deal with the string midpoint where the left and right parts of the string attach. In particular, the BRST operator Q_B becomes rather awkward in this formulation.

Nonetheless, some of the structure of the star product encoded in the three-string vertex is easiest to understand using the half-string formalism, and many formulae related to the 3-string vertex are most easily expressed in terms of the linear map from full-string modes to half-string modes. In this subsection we discuss this linear map, encoded in a matrix X, which we use in subsection 6.4 to give an explicit formulae for the three-string vertex.

Recall that the matter fields are expanded in modes through

$$x(\sigma) = x_0 + \sqrt{2}\sum_{n=1}^{\infty} x_n \cos n\sigma\,. \tag{161}$$

(We suppress Lorentz indices in most of this section for clarity.) We are interested in considering an analogous expansion of the left and right halves of the string. We expand in odd modes with Neumann boundary conditions at the ends of the string, and Dirichlet boundary conditions at the string midpoint:

$$\begin{aligned} l(\sigma) &= x(\sigma) = \sqrt{2}\sum_{k=0}^{\infty} l_{2k+1}\cos(2k+1)\sigma, \quad \sigma < \pi/2 \\ r(\sigma) &= x(\pi-\sigma) = \sqrt{2}\sum_{k=0}^{\infty} r_{2k+1}\cos(2k+1)\sigma, \quad \sigma < \pi/2\,. \end{aligned} \tag{162}$$

Note that there are subtleties associated with the midpoint in this expansion. For example, while we have taken $l(\pi/2)$ to formally vanish, by choosing coefficients like $l_{2k+1} = (-1)^k 2\sqrt{2}a/(2k+1)\pi$ we have $l(\sigma) = a, \forall\sigma < \pi/2$, so $\lim_{\sigma\to\pi/2_-} = a$. These subtleties become important when dealing with the full theory, but are not important in the calculation we carry out below of the three-string vertex.

Let us define the quantities

$$\begin{aligned} X_{2k+1,2n} = X_{2n,2k+1} &= \frac{4(-1)^{k+n}(2k+1)}{\pi\left((2k+1)^2-4n^2\right)} \qquad (n \neq 0)\,, \\ X_{2k+1,0} = X_{0,2k+1} &= \frac{2\sqrt{2}(-1)^k}{\pi(2k+1)}\,. \end{aligned} \tag{163}$$

The matrix

$$X = \begin{pmatrix} 0 & X_{2k+1,2n} \\ X_{2n,2k+1} & 0 \end{pmatrix} \equiv \begin{pmatrix} 0 & X_{oe} \\ X_{eo} & 0 \end{pmatrix} \tag{164}$$

where e, o refer to the set of even and odd indices respectively, is manifestly symmetric and turns out to be orthogonal: $X = X^T = X^{-1}$. Performing a Fourier decomposition we can relate the full-string and half-string modes through

$$\begin{aligned} x_{2n+1} &= \tfrac{1}{2}\left(l_{2n+1} - r_{2n+1}\right)\,, \\ x_{2n} &= \tfrac{1}{2}\sum_{k=0}^{\infty} X_{2n,2k+1}\left(l_{2k+1} + r_{2k+1}\right)\,, \end{aligned} \tag{165}$$

We can invert (165) to derive

$$\begin{aligned} l_{2k+1} &= x_{2k+1} + \sum_{n=0}^{\infty} X_{2k+1,2n}x_{2n}\,, \\ r_{2k+1} &= -x_{2k+1} + \sum_{n=0}^{\infty} X_{2k+1,2n}x_{2n}\,. \end{aligned} \tag{166}$$

One must be careful with the order of summation in sequences of coefficients which do not go to zero faster than $1/n$: different orders of summation may give different results. Fortunately, such associativity anomalies are not relevant for the calculations we do here.

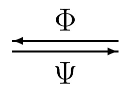
6.3 The two-string vertex $\langle V_2|$

We can immediately apply the oscillator formulae from subsection 6.1 to calculate the two-string vertex. Using the mode decomposition (161), we associate the string field functional

$\Psi[x(\sigma)]$ with a function $\Psi(\{x_n\})$ of the infinite family of string oscillator mode amplitudes. The overlap integral combining (60) and (59) can then be expressed in modes as

$$\int \Psi \star \Phi = \int \prod_{n=0}^{\infty} dx_n dy_n \, \delta(x_n - (-1)^n y_n) \Psi(\{x_n\}) \Phi(\{y_n\}) \,. \tag{167}$$

Geometrically this just encodes the overlap condition $x(\sigma) = y(\pi - \sigma)$ described through



It follows from (156) that we can write the two-string vertex as the squeezed state

$$\langle V_2|_{\text{matter}} = (\langle 0| \otimes \langle 0|) \exp\Big(\sum_{n,m=0}^{\infty} -a_n^{(1)} C_{nm} a_m^{(2)} \Big) \,, \tag{168}$$

where $C_{nm} = \delta_{nm}(-1)^n$ is an infinite-size matrix connecting the oscillator modes of the two single-string Fock spaces, and the sum is taken over all oscillator modes, including zero. In the expression (168), we have used the formalism in which $|0\rangle$ is the vacuum annihilated by a_0. To translate this expression into a momentum basis, we use only $n, m > 0$, and replace

$$(\langle 0| \otimes \langle 0|) \exp\left(-a_0^{(1)} a_0^{(2)} \right) \to \int d^{26}p \, (\langle 0; p| \otimes \langle 0; -p|) \,. \tag{169}$$

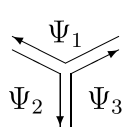
The extension of this analysis to ghosts is straightforward. For the ghost and antighost respectively, the overlap conditions corresponding with $x_1(\sigma) = x_2(\pi - \sigma)$ are [62] $c_1(\sigma) = -c_2(\pi - \sigma)$ and $b_1(\sigma) = b_2(\pi - \sigma)$. This leads to the overall formula for the two-string vertex

$$\begin{aligned} \langle V_2| &= \int d^{26}p \, (\langle 0; p| \otimes \langle 0; -p|) \, (c_0^{(1)} + c_0^{(2)}) \\ &\quad \times \exp\left(-\sum_{n=1}^{\infty} (-1)^n [a_n^{(1)} a_n^{(2)} + c_n^{(1)} b_n^{(2)} + c_n^{(2)} b_n^{(1)}] \right) \,. \end{aligned} \tag{170}$$

This expression for the two-string vertex can also be derived directly from the conformal field theory approach, computing the two-point function of an arbitrary pair of states on the disk.

6.4 The three-string vertex $|V_3\rangle$

The three-string vertex, which is associated with the three-string overlap



can be computed in a very similar fashion to the two-string vertex above. The details of the calculation, however, are significantly more complicated. In this subsection we follow the original approach of Gross and Jevicki [62]; similar approaches were taken by other authors [69, 70]. The method used by Gross and Jevicki is essentially the method used in (156) to write a delta function of two variables in oscillator form by imposing the constraints (159) on a general squeezed state. The 3-string vertex can also be computed by explicitly performing [71, 72] the relevant Gaussian integrals.[1]

From the general structure of the overlap conditions it is clear that, like the two-string vertex, the three-string vertex takes the form of a squeezed state:

$$\begin{aligned}|V_3\rangle = \kappa \int d^{26}p^{(1)}d^{26}p^{(2)}d^{26}p^{(3)} \\ \times \exp\Big(-\frac{1}{2}\sum_{r,s=1}^{3}[a_m^{(r)}V_{mn}^{rs}a_n^{(s)} + 2a_m^{(r)}V_{m0}^{rs}p^{(s)} + p^{(r)}V_{00}^{rs}p^{(s)} + c_m^{(r)}X_{mn}^{rs}b_n^{(s)}]\Big)\,, \\ \times\delta(p^{(1)}+p^{(2)}+p^{(3)})c_0^{(1)}c_0^{(2)}c_0^{(3)}\left(|0;p^{(1)}\rangle\otimes|0;p^{(2)}\rangle\otimes|0;p^{(3)}\rangle\right)\end{aligned} \tag{171}$$

where $\kappa = \mathcal{N} = K^3 = 3^{9/2}/2^6$, and where the Neumann coefficients V_{mn}^{rs} and X_{mn}^{rs} are constants. Writing the momentum basis states in oscillator form

$$|p\rangle = \frac{1}{(\pi)^{1/4}}\exp\left[-\frac{1}{2}p^2 + \sqrt{2}a_0^\dagger p - \frac{1}{2}(a_0^\dagger)^2\right]|0\rangle, \tag{172}$$

we can write matter part of the 3-string vertex as

$$|V_3\rangle = \left(\frac{2\,\pi^{1/4}}{\sqrt{3}(1+V_{00})}\right)^{26}\exp\Big(-\frac{1}{2}\sum_{r,s\leq 3}\sum_{m,n\geq 0}V_{mn}'^{rs}(a_m^{(r)\dagger}\cdot a_n^{(s)\dagger})\Big)\left(|0\rangle\otimes|0\rangle\otimes|0\rangle\right)\,, \tag{173}$$

where $V_{00} = V_{00}^{rr}$ and

$$\begin{aligned}V_{mn}'^{rs} &= V_{mn}^{rs} - \frac{1}{1+V_{00}}\sum_t V_{m0}^{rt}V_{0n}^{ts} \\ V_{m0}'^{rs} &= V_{0m}'^{sr} = \frac{\sqrt{2}}{1+V_{00}}V_{m0}^{rs} \\ V_{00}'^{rs} &= \frac{2}{3(1+V_{00})} + \delta^{rs}\left(1 - 2/(1+V_{00})\right)\,.\end{aligned} \tag{174}$$

We now want to determine the coefficients $V_{mn}'^{rs}$ by using overlap conditions analogous to (159). It is convenient to use a $\mathbf{Z}_3$ Fourier decomposition of the three string modes $x^{(i)}(\sigma)$

$$\begin{aligned}Q &= \frac{1}{\sqrt{3}}(x^{(1)} + \omega x^{(2)} + \omega^2 x^{(3)}) \\ Q^{(3)} &= \frac{1}{\sqrt{3}}(x^{(1)} + x^{(2)} + x^{(3)})\end{aligned} \tag{175}$$

[1] Another approach to the cubic vertex has been explored extensively. By diagonalizing the Neumann matrices, the star product takes the form of a continuous Moyal product [7, 73]. This simplifies the vertex but complicates the propagator. For a recent discussion of this work, applications of this approach, and further references, see the review of Bars [9].

where $\omega = \exp(2\pi i/3)$. The definitions (175) can be used to define $Q(\sigma)$ in terms of $x^{(i)}(\sigma)$ as well as to define Q_n in terms of $x_n^{(i)}$ (and similarly for $Q^{(3)}$); henceforth by Q we denote the collection of full-string modes Q_n (and similarly for $Q^{(3)}$). We can relate the full-string modes $Q, Q^{(3)}$ to half-string modes $L, R, L^{(3)}, R^{(3)}$ through the equations (166). In terms of these variables, the overlap conditions are

$$\begin{aligned} L - \omega R &= 0, \\ L^{(3)} - R^{(3)} &= 0\,. \end{aligned} \tag{176}$$

In terms of the even and odd full-string modes (which, using the same notation as in section 6.2, we denote by $Q_{e,o}$) these conditions are expressed as

$$Q_o - i\sqrt{3}X_{oe}Q_e = 0\,, \tag{177}$$

and

$$Q_o^{(3)} = 0\,. \tag{178}$$

Multiplying by X_{eo}, (177) can be rewritten as

$$\frac{i}{\sqrt{3}}X_{eo}Q_o + Q_e = 0\,. \tag{179}$$

This can be combined with (177) and written in the simpler form

$$(1 - Y)Q = 0, \tag{180}$$

where

$$Y = -\frac{1}{2}C + \frac{\sqrt{3}}{2}X\,. \tag{181}$$

Note that $Y^2 = 1$. Similarly, we can write (178) as

$$(1 - C)Q^{(3)} = 0\,. \tag{182}$$

Equations (180) and (182) are the essential overlap equations satisfied by the three-string vertex. Writing the three-string vertex as a squeezed state in terms of oscillators $A, A^{(3)}$ related to the string oscillators $a^{(i)}$ through the analogue of (175), we then have

$$|V_3\rangle \sim \exp\left(-A^\dagger U \bar{A}^\dagger - \frac{1}{2}(A^{(3)})^\dagger C (A^{(3)})^\dagger\right)\,, \tag{183}$$

where U satisfies the overlap constraint (180). Recall that the string modes are proportional to

$$x \sim E(a - a^\dagger) \tag{184}$$

where

$$E_{mn} = \delta_{mn}\frac{1}{\sqrt{m}}, \quad m \neq 0, \qquad E_{00} = 1/\sqrt{2}\,. \tag{185}$$

Thus, from (180) and (183) we see that U must satisfy the overlap constraint

$$(1-Y)E(1+U)=0\,. \tag{186}$$

As we discussed in the last part of subsection 6.1, associated with this constraint there is an analogous constraint on the derivatives in the perpendicular direction. Since $Y^2=1$, we have

$$(1+Y)(1-Y) \;=\; (1-Y)(1+Y)=0\,. \tag{187}$$

Since derivatives with respect to the x modes go as

$$\partial \sim E^{-1}(a+a^\dagger), \tag{188}$$

we have the additional overlap constraint on U

$$(1+Y)E^{-1}(1-U)=0\,. \tag{189}$$

Equations (186) and (189) determine U completely, giving

$$U=(2-EYE^{-1}+E^{-1}YE)[EYE^{-1}+E^{-1}YE]^{-1}. \tag{190}$$

Unfortunately, the matrix combination in brackets is difficult to explicitly invert. This does, however, give a closed form expression for the three-string vertex (173), where

$$\begin{aligned} V'^{rr} &= \frac{1}{3}(C+U+\bar{U}) \\ V'^{r,r\pm1} &= \frac{1}{6}(2C-U-\bar{U}) \pm \frac{i\sqrt{3}}{6}(U-\bar{U})\,. \end{aligned} \tag{191}$$

While (190) is difficult to directly compute, given a formula for U one can check that the formula is correct by checking the overlap conditions (186) and (189). Expressions for V and X and hence for U and V' were computed [62] by essentially the method used in the previous section. Their results for V and X are given as follows[2]. Define A_n, B_n for $n \geq 0$ through

$$\begin{aligned} \left(\frac{1+ix}{1-ix}\right)^{1/3} &= \sum_{n\,\mathrm{even}} A_n x^n + i\sum_{m\,\mathrm{odd}} A_m x^m \\ \left(\frac{1+ix}{1-ix}\right)^{2/3} &= \sum_{n\,\mathrm{even}} B_n x^n + i\sum_{m\,\mathrm{odd}} B_m x^m\,. \end{aligned} \tag{192}$$

[2]Note that in some references, signs and various factors in κ and the Neumann coefficients may be slightly different. In some papers, the cubic term in the action is taken to have an overall factor of $g/6$ instead of $g/3$; this choice of normalization gives a 3-tachyon amplitude of g instead of $2g$, and gives a different value for κ. Often, the sign in the exponential of (171) is taken to be positive, which changes the signs of the coefficients V^{rs}_{nm}, X^{rs}_{nm}. When the matter Neumann coefficients are defined with respect to the oscillator modes α_n rather than a_n, the matter Neumann coefficients V^{rs}_{nm}, V^{rs}_{n0} must be divided by $\sqrt{nm}$ and $\sqrt{n}$. This is the case for the coefficients N^{rs}_{nm} computed in (140), which are related to the V's through $N^{rs}_{nm}=V^{rs}_{nm}/\sqrt{nm}$. Finally, when α' is taken to be $1/2$, an extra factor of $1/\sqrt{2}$ appears for each 0 subscript in the matter Neumann coefficients.

These coefficients can be used to define 6-string Neumann coefficients $N^{r,\pm s}_{nm}$ through

$$\begin{aligned} N^{r,\pm r}_{nm} &= \begin{cases} \frac{1}{3(n\pm m)}(-1)^n(A_nB_m \pm B_nA_m), & m+n\,\text{even},\ m\neq n \\ 0, \quad m+n\,\text{odd} \end{cases} \\ N^{r,\pm(r+\sigma)}_{nm} &= \left\{ \begin{array}{l} \frac{1}{6(n\pm\sigma m)}(-1)^{n+1}(A_nB_m \pm \sigma B_nA_m), \quad m+n\,\text{even},\ m\neq n \\ \sigma\frac{\sqrt{3}}{6(n\pm\sigma m)}(A_nB_m \mp \sigma B_nA_m), \quad m+n\,\text{odd} \end{array} \right] . \end{aligned} \tag{193}$$

where in $N^{r,\pm(r+\sigma)}$, $\sigma = \pm 1$, and $r+\sigma$ is taken modulo 3 to be between 1 and 3. The 3-string matter Neumann coefficients V^{rs}_{nm} are then given by

$$\begin{aligned} V^{rs}_{nm} &= -\sqrt{mn}(N^{r,s}_{nm} + N^{r,-s}_{nm}), \quad m\neq n,\ \text{and}\, m,n\neq 0 \\ V^{rr}_{nn} &= -\frac{1}{3}\left[2\sum_{k=0}^{n}(-1)^{n-k}A_k^2 - (-1)^n - A_n^2\right], \quad n\neq 0 \\ V^{r,r+\sigma}_{nn} &= \frac{1}{2}\left[(-1)^n - V^{rr}_{nn}\right], \quad n\neq 0 \\ V^{rs}_{0n} &= -\sqrt{2n}\left(N^{r,s}_{0n} + N^{r,-s}_{0n}\right), \quad n\neq 0 \\ V^{rr}_{00} &= \ln(27/16) \end{aligned} \tag{194}$$

The ghost Neumann coefficients $X^{rs}_{mn}, m\geq 0, n>0$ are given by

$$\begin{aligned} X^{rr}_{mn} &= \left(-N^{r,r}_{nm} + N^{r,-r}_{nm}\right), \quad n\neq m \\ X^{r(r\pm 1)}_{mn} &= m\left(\pm N^{r,r\mp 1}_{nm} \mp N^{r,-(r\mp 1)}_{nm}\right), \quad n\neq m \\ X^{rr}_{nn} &= \frac{1}{3}\left[-(-1)^n - A_n^2 + 2\sum_{k=0}^{n}(-1)^{n-k}A_k^2 - 2(-1)^nA_nB_n\right] \\ X^{r(r\pm 1)}_{nn} &= -\frac{1}{2}(-1)^n - \frac{1}{2}X^{rr}_{nn} \end{aligned} \tag{195}$$

These expressions for the matter and ghost Neumann coefficients were computed by Gross and Jevicki [62], and include minor corrections published later [74]. It was shown that the resulting matter matrices U indeed satisfy the overlap conditions (186) and (189). This shows that the conformal field theory method and the oscillator method give the same results for the matter part of the three-string vertex. The same is true for the ghost part of the vertex, although we will not go into the details of this discussion here.

Before leaving the three-string vertex, it is worth noting that the Neumann coefficients have a number of simple symmetries. There is a cyclic symmetry under $r\to r+1, s\to s+1$, which corresponds to the obvious geometric symmetry of rotating the vertex. The coefficients are also symmetric under the exchange $r\leftrightarrow s, n\leftrightarrow m$. Finally, there is a twist symmetry which, as discussed in section 4.3, is associated with reflection of the strings

$$\begin{aligned} V^{rs}_{nm} &= (-1)^{n+m}V^{sr}_{nm} \\ X^{rs}_{nm} &= (-1)^{n+m}X^{sr}_{nm} . \end{aligned} \tag{196}$$

This symmetry follows from the fact that half-strings carrying odd modes pick up a minus sign under reflection. Since each string carrying an odd mode gets two changes of sign, from the two ends of the string, it is straightforward to see that this symmetry guarantees that the three-vertex is invariant under reflection, and therefore satisfies condition (76).

6.5 Calculating the SFT action

Given the action

$$S=-\frac{1}{2}\langle V_2|\Psi,Q\Psi\rangle-\frac{g}{3}\langle V_3|\Psi,\Psi,\Psi\rangle\,, \tag{197}$$

and the explicit formulae (170, 171) for the two- and three-string vertices, we can in principle calculate the string field action term by term for each of the fields in the string field expansion

$$\begin{aligned}\Psi \;=\; \int d^{26}p\;\big[\phi(p)\,|0_1;p\rangle+A_\mu(p)\,\alpha_{-1}^\mu|0_1;p\rangle+\chi(p)b_{-1}c_0|0_1;p\rangle\\ +B_{\mu\nu}(p)\alpha_{-1}^\mu\alpha_{-1}^\nu|0_1;p\rangle+\cdots\big]\,.\end{aligned} \tag{198}$$

Since the resulting action has an enormous gauge invariance given by (54), it is often helpful to fix the gauge before computing the action. A particularly useful gauge choice is the Feynman-Siegel gauge

$$b_0|\Psi\rangle=0\,. \tag{199}$$

This is a good gauge choice locally, fixing the linear gauge transformations $\delta|\Psi\rangle=Q|\Lambda\rangle$. This gauge choice is not, however, globally valid; we will return to this point in subsection 7.4. In this gauge, all fields in the string field expansion which are associated with states that have an antighost zero-mode c_0 are taken to vanish. For example, the field $\chi(p)$ in (198) vanishes. In Feynman-Siegel gauge, the BRST operator takes the simple form

$$Q=c_0L_0=c_0(N+p^2-1) \tag{200}$$

where N is the total (matter + ghost) oscillator number.

Using (200), it is straightforward to write the quadratic terms in the string field action. They are

$$\frac{1}{2}\langle V_2|\Psi,Q\Psi\rangle=\int d^{26}p\;\left\{\phi(-p)\left[\frac{p^2-1}{2}\right]\phi(p)+A_\mu(-p)\left[\frac{p^2}{2}\right]A^\mu(p)+\cdots\right\}\,. \tag{201}$$

The cubic part of the action can also be computed term by term, although the terms are somewhat more complicated. The leading terms in the cubic action are given by

$$\begin{aligned}\frac{1}{3}\langle V_3|\Psi,\Psi,\Psi\rangle=\int d^{26}p\,d^{26}q\;\frac{\kappa g}{3}\;e^{(\ln 16/27)(p^2+q^2+p\cdot q)}\\ \times\Big\{\phi(-p)\phi(-q)\phi(p+q)+\frac{16}{9}A^\mu(-p)A_\mu(-q)\phi(p+q)\\ -\frac{8}{9}(p^\mu+2q^\mu)(2p^\nu+q^\nu)A^\mu(-p)A_\nu(-q)\phi(p+q)+\cdots\Big\}\,.\end{aligned} \tag{202}$$

In computing the ϕ^3 term we have used

$$V_{00}^{rs}=\delta^{rs}\ln(\frac{27}{16})\,. \tag{203}$$

The $A^2\phi$ term uses

$$V_{11}^{rs} = -\frac{16}{27}, \quad r \neq s, \tag{204}$$

while the $(A \cdot p)^2\phi$ term uses

$$V_{10}^{12} = -V_{10}^{13} = -\frac{2\sqrt{2}}{3\sqrt{3}}\,. \tag{205}$$

The most striking feature of this action is that for a generic set of three fields, there is a *nonlocal* cubic interaction term that contains an exponential of a quadratic form in the momenta. This means that the target space formulation of string theory has a dramatically different character from a standard quantum field theory. From the point of view of quantum field theory, string field theory seems to contain an infinite number of nonrenormalizable interactions. Just like the simpler case of noncommutative field theories, however, the magic of string theory seems to combine this infinite set of interactions into a sensible model. It has been shown that all on-shell amplitudes computed from the string field theory action we have discussed here precisely reproduce the amplitudes given by the usual conformal field theory approach, including the correct measure on moduli space [75, 76, 77]. Note, though, that the bosonic open theory becomes problematic at the quantum level because of the closed string tachyon, whose instability is not yet understood. For the purposes of these lectures, we will restrict our attention to the classical bosonic open string action. Open superstring field theory should be better behaved since the closed string sector has no tachyon. There has been significant progress in understanding tachyon condensation in superstring field theory [78, 37], even though superstring field theory is less developed than bosonic string field theory.

7 Evidence for the Sen conjectures

In this section we review the evidence from Witten's OSFT for Sen's conjectures. Subsection 7.1 contains a derivation of the formula for the tension of a bosonic D-brane. In subsection 7.2, a general discussion is given of symmetries in the string field theory action and resulting constraints on the set of string fields which take nonzero values in the tachyon vacuum. Subsection 7.3 contains a summary of existing results for the determination of the stable vacuum in Witten's OSFT (Sen's first conjecture), including some results which appeared after these lectures were originally given in 2001. In subsection 7.4 we discuss the Feynman-Siegel gauge choice and its limitations. Subsection 7.5 summarizes results on lower-dimensional D-branes as solitons in OSFT (Sen's second conjecture). Subsection 7.6 discusses the general problem of finding all open string backgrounds within OSFT. Sen's third conjecture is discussed in the following section 8, which is completely devoted to a discussion of the physics in the stable vacuum (vacuum string field theory).

7.1 Tension of bosonic Dp-branes

In this section we learn how to relate the open string coupling constant of string field theory to the mass of the D-brane described by the open string field theory. The material presented in this subsection is an unpublished result due to Ashoke Sen [79], who cited the result in the paper [39] on the subject of universality. Subsequently, the closely related computation for the superstring was explained in detail [80]. For an alternative check of the result, see Appendix A of the paper by Okawa [81].

In general, an open string field theory is formulated using a BCFT (boundary conformal field theory) which describes some D-brane (or a configuration of D-branes). In order to describe a D-brane with finite mass, we consider a compactification of p spatial coordinates and wrap a Dp-brane around along these dimensions. The string field theory associated with this D-brane is written as before:

$$S(\Phi) = -\frac{1}{g^2}\left[\frac{1}{2}\langle\,\Phi\,,\,Q\,\Phi\rangle + \frac{1}{3}\langle\,\Phi\,,\,\Phi * \Phi\rangle\right]. \tag{206}$$

The D-brane in question is perceived by the effective $(25-p)$-dimensional observer as a point particle. The BCFT includes a Neumann field X^0, a set of Dirichlet fields X^i, with $i = 1, \ldots, 25-p$ and some set of Neumann fields X^a, with $a = 25-p+1, \ldots, 25$ that describe the internal sector of the BCFT. The string field theory effectively describes an infinite collection of fields $\phi_i(t, x^a)$. These fields do not depend on $x^1, \ldots, x^{25-p}$ because the corresponding string coordinates are Dirichlet. Since the coordinates x^a are compact, the fields $\phi_i(t, x^a)$ can be expanded in Fourier modes. These are a collection of degrees of freedom that are just time dependent. The string field theory action then reduces to an integral over time of a time-dependent Lagrangian density.

We will set up the string field theory in such a way that all dimensions (including time) are compactified on circles of unit circumference. In this case, the mass M of the Dp-brane coincides with the tension of the Dp-brane. The claim is that

$$M = \frac{1}{2\pi^2\,g^2}\,. \tag{207}$$

In this formula and in the following, we set $\alpha' = 1$. In these units the string tension is $T_0 = 1/(2\pi)$. When we consider the string field theory of a D25-brane, (207) gives

$$T_{25} = \frac{1}{2\pi^2\,g^2}\,. \tag{208}$$

We begin our study by considering some special momentum states of the BCFT:

$$|k_0\rangle \equiv e^{(ik_0 X^0(0))}|0\rangle\,. \tag{209}$$

Moreover, we will normalize these states by declaring

$$\langle k_0|\,c_{-1}c_0c_1\,|k_0'\rangle = \delta_{k_0,k_0'}\,, \tag{210}$$

consistent with the discussion below (98). Since the time direction has been made compact via $t \sim t+1$, the time component k_0 of the momentum is quantized: $k_0 = 2\pi n$, with n integer, and we can use a Kronecker delta in the above inner product.

We will consider the computation of the brane mass in three steps.

Step 1: We consider time-dependent displacements of the D-brane. We will write down a string field that describes such a displacement and evaluate the kinetic term of the string action. This will make it clear how we can hope to calculate the brane mass.

Let X^i be one of the Dirichlet directions for the D-brane and assume that $x^i = 0$ is the original position of the brane. Consider now a displacement field $\phi^i(t)$ that is expected to be proportional to a coordinate displacement $x^i(t)$. We expand the field $\phi^i(t)$ as:

$$\phi^i(t) = \sum_{k_0} e^{ik_0 t}\, \phi^i(k_0)\,, \tag{211}$$

and we use the Fourier components $\phi^i(k_0)$ to assemble the corresponding string field:

$$|\Phi\rangle = \sum_{k_0} \phi^i(k_0)\, c_1 \alpha^i_{-1}\, |k_0\rangle\,. \tag{212}$$

As you can see, the string field is built using states of the massless scalar field that represents translations of the D-brane. For this string field, the kinetic term $S_2(\Phi)$ of the string action is given by

$$S_2(\Phi) = -\frac{1}{g^2} \sum_{k_0, k_0'} \phi^i(k_0)\phi^i(k_0')\langle -k_0'|c_{-1}\alpha^i_1\, c_0 L_0\, c_1\alpha^i_{-1}|k_0\rangle\,. \tag{213}$$

Since $L_0 = p^2 + \ldots$ where the terms indicated by dots vanish in the present case, $L_0 = -k_0^2$ in (213) and

$$S_2(\Phi) = \frac{1}{2g^2} \sum_{k_0} \phi^i(-k_0)\, k_0^2 \phi^i(k_0)\,. \tag{214}$$

Let us now rewrite this string action in terms of the field $\phi^i(t)$ introduced in (211). A short computation gives

$$\int_0^1 dt\, \partial_t \phi^i\, \partial_t \phi^i = \sum_{k_0} \phi^i(-k_0)\, k_0^2 \phi^i(k_0)\,. \tag{215}$$

Comparing with (214) we find that

$$S_2(\Phi) = \frac{1}{2g^2} \int_0^1 dt\, \partial_t \phi^i\, \partial_t \phi^i\,. \tag{216}$$

As we mentioned earlier, the field $\phi^i(t)$ is expected to be proportional to the position $x^i(t)$ of the brane (at least for small, slowly varying displacements), so we can rewrite the above action as

$$S_2(\Phi) = \frac{1}{g^2}\Big(\frac{\delta\phi^i}{\delta x^i}\Big)^2 \int_0^1 dt\ \frac{1}{2}\partial_t x^i\, \partial_t x^i \tag{217}$$

where the derivatives are evaluated at zero displacement. Since $\partial_t x^i$ is the velocity of the D-brane, the above action represents the contribution from the (non-relativistic) kinetic energy of a D-brane that has a mass M given by [3]

$$M = \frac{1}{g^2}\Big(\frac{\delta\phi^i}{\delta x^i}\Big)^2 . \tag{218}$$

Step 2. To find out how $\delta\phi^i$ is related to a true displacement δx^i, we add a reference D-brane a distance b away from our original brane, in the direction x^i. We will then consider a string stretched between the branes. We will use the string field action to compute the change in the mass of such string when our D-brane is displaced by some $\delta\phi^i$. Since the string tension is known, we will be able to calculate the value of the physical displacement δx^i.

Given a string of length L, its mass includes a contribution $T_0 L = L/(2\pi)$, and the corresponding contribution to the mass-squared is $L^2/(4\pi^2)$. If the original stretched string has length b and its length is then changed to $b + \delta x^i$, the change δm^2 of the mass-squared is

$$\delta m^2 = \frac{1}{4\pi^2}\Big((b+\delta x^i)^2 - b^2\Big) \simeq \frac{1}{2\pi^2}\, b\,\delta x^i . \tag{219}$$

Let us now consider a time independent displacement, that is, a configuration with $k_0 = 0$ (see (211)). We thus set $\delta\phi^i \equiv \delta\phi^i(k_0 = 0)$ and $\delta\phi^i(k_0 \neq 0) = 0$. The string field associated to this displacement is obtained using (212):

$$|\delta\Phi\rangle = \delta\phi^i\, c_1\alpha^i_{-1}\,|0\rangle . \tag{220}$$

We want to learn the effect of this string field perturbation on the masses of stretched strings. To do so, we introduce a complex field η. The fields η and η^* represent the string that stretches from our brane (brane one) to the reference brane (brane two) and the string that stretches from the reference brane to our brane, respectively. The string field that describes these states is written as:

$$|\psi\rangle = \eta\, c_1|k_0, b\rangle \otimes \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} + \eta^*\, c_1| - k_0, -b\rangle \otimes \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} . \tag{221}$$

The matrices included here are Chan-Paton matrices $a_{\alpha\beta}$, with $\alpha, \beta = 1, 2$. A value of one for a given $a_{\alpha\beta}$, with zero for all other entries, is used to represent a string that stretches from brane α to brane β. When we have parallel D-branes, the string field is matrix-valued. The string action includes a trace operation Tr that applies to the matrices, and the star product includes matrix multiplication. The state $|k_0, b\rangle$ represents the ground state of a string with momentum k_0 that stretches a distance b in the x^i direction. It is necessary for

[3] Note that at this point, it is possible to take a shortcut to get the D-brane mass directly using the fact that SFT at tree level reproduces Yang-Mills theory [82], with $g_{\rm YM} = g/\sqrt{2}$ [43, 82], where the Yang-Mills field appears in the string field expansion as $A_\mu(k)\alpha^\mu_{-1}|0_1; k\rangle$, and where an additional factor of 2π arises from the T-duality relation from section 2.4, $X \to 2\pi\alpha' A$. Thus, replacing $A^i \to X^i/2\pi$ in the Yang-Mills action we have $1/2g^2_{\rm YM}F_{0i}F^{0i} \to 1/2g^2(\partial_0 x^i/\sqrt{2}\pi)^2$, so $M = 1/2\pi^2 g^2$. Although perhaps more transparent, this is essentially the same calculation as the original argument of Sen presented here, which we include in full as it sheds light on the structure of the theory.

our analysis to determine the CFT vertex operator that corresponds to this stretched string. We claim that the operator is

$$|k_0, b\rangle \quad \longleftrightarrow \quad e^{ik_0X^0}\, e^{i\frac{b}{2\pi}(X_L^i - X_R^i)}\,. \tag{222}$$

The k_0 dependence of the operator is already familiar from (209). The field multiplying b is formed from the left-moving and right-moving parts of the open string coordinate X^i, evaluated at the string endpoint. This coordinate X^i satisfies Dirichlet boundary conditions, so at the boundary $X_L^i = -X_R^i$, and we can replace $X_L^i - X_R^i$ by $2X_L^i$. We also have the operator products and stress tensor

$$\partial X_L^i(x)\, \partial X_L^i(y) \sim -\frac{1}{2}\frac{1}{(x-y)^2}\,, \qquad T_{X^i} = -\partial X_L^i \partial X_L^i\,. \tag{223}$$

These relations allow us to compute the conformal dimension of an exponential. One readily finds that $\exp(i\alpha X_L^i)$ has conformal dimension $\alpha^2/4$. It follows that

$$\text{dimension}\,\Big(e^{i\frac{b}{\pi}X_L^i}\Big) = \Big(\frac{b}{2\pi}\Big)^2\,. \tag{224}$$

Since conformal dimension is the value of L_0, which, in turn, determines the mass-squared, this result confirms that the operator in (222) has correctly reproduced the mass-squared of the stretched string. For future use, the vertex operator can be written as

$$e^{ik_0X^0}\, e^{i\frac{b}{\pi}X_L^i}\,. \tag{225}$$

The evaluation of the kinetic term for the field in (221) is relatively straightforward. The only terms that survive are the off-diagonal ones, coupling η and η^*. There are two such terms, and their contributions are identical. The product of the two matrices give a matrix of trace one, and the overlap $\langle -k_0, -b|k_0, b\rangle$ is also equal to one. We then find

$$g^2\, S_2(\eta, \eta^*) = -\frac{1}{2}\cdot 2\,\eta^*\Big(-k_0^2 + \frac{b^2}{(2\pi)^2}\Big)\eta = \eta^*\Big(k_0^2 - \frac{b^2}{(2\pi)^2}\Big)\eta\,. \tag{226}$$

In the setup with two branes, the fluctuation (220) that represents the displacement of our brane is fully represented by

$$|\delta\Phi\rangle = \delta\phi^i\, c_1\alpha_{-1}^i\,|0\rangle \otimes \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\,. \tag{227}$$

With the chosen normalization for X^i, the vertex operator associated with $\alpha_{-1}^i|0\rangle$ is $i\sqrt{2}\partial X_L$.

Step 3. We must now include the effects of the interactions to see how the fluctuation (227) affects the mass of the stretched string. Since the mass can be read from equation (226), we will find a term proportional to $\eta^*\eta$ that arises from the interaction and modifies the value of the mass.

The interaction term takes the form

$$g^2 S_3(\Phi) = -\frac{1}{3}\langle \Phi, \Phi, \Phi\rangle\,, \tag{228}$$

and the string field is taken to be $|\Phi\rangle = |\psi\rangle + |\delta\Phi\rangle$, in order to see the effect of the fluctuation on the stretched string. We are looking for the terms of the form $\eta^*\eta\,\delta\phi^i$, so we have three different operators to insert at three different punctures. There are a total of six possible arrangements, that can be divided into two groups of three arrangements each. In each of these groups the cyclic ordering of the operators is the same. The Chan-Paton matrices imply that one cyclic ordering contributes while the other does not. Indeed,

$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \tag{229}$$

is a matrix of unit trace, while

$$\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \tag{230}$$

is a matrix of vanishing trace. We conclude that the Chan-Paton matrices contribute a factor of $+3$. The operators to be inserted can be chosen to be physical (dimension zero) so we need not worry about local coordinates at the punctures. Using punctures at ∞, -1, and 0 we then find:

$$g^2 S_3(\Phi) = -\eta^*\eta\delta\phi^i \Big\langle e^{-ik_0X^0 - i\frac{b}{\pi}X_L^i} c(\infty)\,\sqrt{2}\,i\,\partial X_L c(-1)\, e^{ik_0X^0 + i\frac{b}{\pi}X_L^i} c(0)\Big\rangle\,. \tag{231}$$

Since the vertex operators are on-shell, and the ghost insertions are placed at standard positions, the whole correlator gives a factor of one, except for the contraction between the $\partial X_L(-1)$ and the finitely located $\exp(i\frac{b}{\pi}X_L^i(0))$:

$$g^2 S_3 = -\eta^*\eta\delta\phi^i\,\sqrt{2}\,i\,\frac{ib}{\pi}\Big(-\frac{1}{2}\Big)\frac{1}{(-1-0)} = \frac{b}{\sqrt{2}\,\pi}\eta^*\eta\delta\phi^i\,. \tag{232}$$

Combining this result with (226) we find

$$g^2\,(S_2 + S_3) = \eta^*\Big(k_0^2 - \frac{b^2}{(2\pi)^2} + \frac{b}{\sqrt{2}}\frac{\delta\phi^i}{\pi}\Big)\eta\,. \tag{233}$$

The last term in parenthesis corresponds to a change in m^2. So, comparing with (219) we obtain

$$\frac{1}{2\pi^2} b\,\delta x^i = -\frac{b}{\sqrt{2}}\frac{\delta\phi^i}{\pi} \quad \rightarrow \quad \frac{\delta\phi^i}{\delta x^i} = -\frac{1}{\sqrt{2}\,\pi}\,. \tag{234}$$

This is the needed relation between the field $\delta\phi^i$ that represents a displacement of the brane and the resulting displacement δx^i. The mass of the brane now follows directly from (218):

$$M = \frac{1}{g^2}\Big(\frac{1}{\sqrt{2}\,\pi}\Big)^2 = \frac{1}{2\pi^2 g^2}\,. \tag{235}$$

This is the result we wanted to establish.

7.2 Constraints and symmetries

It may appear that *a priori* all scalar fields in the spectrum of open strings could acquire a vacuum expectation value in the tachyonic vacuum. Nevertheless, there are a set of considerations that imply that only a subset of these scalar fields acquire expectation values. In this section we explore these ideas. They are subdivided into the following:

(1) Universality conditions.

(2) Twist conditions.

(3) Gauge fixing conditions.

(4) $SU(1,1)$ conditions.

Among these conditions, the third one, which concerns gauge fixing, is on a somewhat different footing. The other three conditions apply because of a simple general argument which we discuss first.

Consider a subdivision of all the scalar fields into two disjoint set of fields. The first set contains the fields $t_0, t_1, t_2, \ldots$ and the second set contains the fields $u_0, u_1, u_2, \ldots$. Let us denote by t_i the elements of the first set and by u_a the elements of the second set. Suppose the string field action $S(t_i, u_a)$ is such that there are no terms that are linear in u_a. We then claim that it is consistent to search for a solution of the equations of motion that assumes $u_a = 0$ for all a. The reason is easy to explain. If all terms with u fields contain at least two of them, the equations of motion for the u fields are composed of terms all of which contain at least one u field. As a result, $u_a = 0$ satisfies these equations of motion. In our analysis we will try to construct a set $\{t_i\}$ with the smallest possible number of fields, so that none of the remaining fields couples linearly in the action. The tachyon field, of course, is one of the elements of the set $\{t_i\}$.

Let's begin by explaining how (1) works. For this we split the state space of the BCFT into three groups. In each of these groups, the ghost part of the states is the same: it includes all states of ghost number one. The nontrivial part of the argument uses the matter part of the conformal field theory. We write

$$\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2 \oplus \mathcal{H}_3 \,, \tag{236}$$

where $\mathcal{H}_1, \mathcal{H}_2$, and $\mathcal{H}_3$ will be disjoint vector subspaces of $\mathcal{H}$ (their intersections give the zero vector). We also write

$$\mathcal{H}_i = \mathcal{M}_i \otimes |\mathcal{G}\rangle \,, \quad i = 1, 2, 3 \,. \tag{237}$$

where $|\mathcal{G}\rangle$ denotes the general state of ghost number one in the ghost conformal field theory. The spaces $\mathcal{M}_1, \mathcal{M}_2$, and $\mathcal{M}_3$ are disjoint subspaces of the matter CFT whose union gives the total matter CFT state space. The $\mathcal{M}$ subspaces are defined as:

$$\begin{aligned}
\mathcal{M}_1 : &\quad \text{primary} \;\; |0\rangle \text{ and descendents,} \\
\mathcal{M}_2 : &\quad \text{primaries } |k_0 \neq 0\rangle \text{ and descendents,} \\
\mathcal{M}_3 : &\quad \text{primaries } |k_0 = 0\rangle \text{ different from } |0\rangle \text{ and descendents}\,.
\end{aligned} \tag{238}$$

In the above, primary means Virasoro primary in the matter sector, and descendent means Virasoro descendent. The union of the spaces give the full CFT because for unitary matter CFT's (which we are assuming our CFT is) the state space can be broken into primaries and their descendents. Any matter primary belongs to one of the three spaces above. It should also be clear that the primaries in $\mathcal{M}_3$ have positive conformal dimension.

We now claim that the fields in $\mathcal{H}_2$ and $\mathcal{H}_3$ need not acquire expectation values (they are u fields); the tachyon condensate is all in $\mathcal{H}_1$. We are therefore defining the t fields to be precisely the fields in $\mathcal{H}_1$. To prove that this is valid, we first note that a field in $\mathcal{H}_2$ cannot appear linearly in a term where all other fields are t fields (*i.e.*, fields in $\mathcal{H}_1$). The reason is simply momentum conservation.

A little more work is necessary to show that the fields in $\mathcal{H}_3$ cannot couple linearly to the fields in $\mathcal{H}_1$. Let us begin with the kinetic term. Since the BRST operator is composed of terms that include ghost oscillators and matter Virasoro operators, it maps each $\mathcal{H}_i$ space into itself. The primaries in $\mathcal{H}_1$ and $\mathcal{H}_3$ are BPZ orthogonal, so any two states in the descendent towers are also orthogonal (this is proven by using the BPZ conjugation properties of Virasoro operators to move them from one state to the other until some state is annihilated or the whole expression reduces to the BPZ inner product of the primaries). For the interaction term a similar argument holds. First note that the three string vertex does not couple two matter primaries from $\mathcal{H}_1$ to a matter primary from $\mathcal{H}_3$. This is because in the CFT matter correlator the primaries from $\mathcal{H}_1$ appear as identity operators, so the whole correlator is proportional to the one-point function of the primary in $\mathcal{H}_3$, which vanishes because the state has non-zero dimension. The Virasoro conservation laws on the vertex then imply that the coupling of any two states in $\mathcal{H}_1$ to a state in $\mathcal{H}_3$ must vanish. This completes our proof.

The space $\mathcal{H}_1$ is universal. It does not depend on the details of the matter conformal field theory, except for the existence of a zero-momentum SL(2,R) ground state. The space can be written as

$$\mathcal{H}_1 \equiv \text{Span}\Big\{ L^m_{-j_1} \dots L^m_{-j_p}\, b_{-k_1} \dots b_{-k_q}\, c_{-l_1} \dots c_{-l_r}\, |0\rangle \Big\} \tag{239}$$

where

$$j_1 \geq j_2 \geq \dots \geq j_p\,, \quad j_i \geq 2\,, \quad k_i \geq 2\,, \quad l_i \geq -1\,, \quad \text{and} \quad l - q = 1\,. \tag{240}$$

The first inequality ensures that the descendents are built unambiguously, the second inequality is needed because $L_{-1}|0\rangle = 0$. The third and fourth inequalities are familiar, and the last equality ensures that the ghost number of the state is one.

Let us now explain how twist properties allow us to restrict $\mathcal{H}_1$ further. The claim is that we can restrict ourselves to the twist even subspace of $\mathcal{H}_1$. Heuristically, this follows from the fact that the two- and three-string vertices are invariant under reflection, so all terms linear in twist fields would pick up a change of sign and therefore vanish. The twist-even space, of course, contains the zero momentum tachyon $c_1|0\rangle$ (recall that $|0\rangle$ is twist odd, and $\Omega c_{-n} \Omega^{-1} = (-)^n c_{-n}$). The first two properties in (75) ensure that the kinetic term in the string action does not couple a twist odd field to a twist even field. We also studied the twist properties of the three string vertex. In fact, in an exercise, we considered a twist even field A_+ and a twist odd field A_- ($\Omega A_\pm = \pm A$) both of which were Grassmann odd (like the string field is). You then showed that $\langle A_+, A_+, A_-\rangle = 0$ (see (79)). Consider now a general string field $\Phi \in \mathcal{H}$ and split it into twist even and twist odd parts $\Phi = \Phi_+ + \Phi_-$. When the interaction vertex is evaluated, the terms linear in Φ_- are of the form $\langle \Phi_+, \Phi_+, \Phi_-\rangle$ (any other similar looking term is related to this by cyclicity). So terms linear on twist odd fields vanish. This proves that we can indeed constrain $\mathcal{H}_1$ further.

The twist eigenvalue of a state is given as $\Omega = (-1)^N$, where N is the number eigenvalue of the state, defined with $N = 0$ for the zero momentum tachyon. In terms of level, states at odd levels are twist odd, and states at even levels are twist even. So, the twist condition allows us to restrict ourselves to the states of $\mathcal{H}_1$ at even levels.

We now turn to the gauge fixing condition. This gauge fixing condition, the Feynman-Siegel gauge condition $b_0|\Phi\rangle = 0$, restricts further the space $\mathcal{H}_1$. We will discuss the global validity of the Siegel gauge later, but here we discuss its clear validity at the linearized level and within the subspace $\mathcal{H}_1$ already restricted to states at even level. First, we show that the gauge condition can be reached starting from fields that do not satisfy it. Let $|\Phi\rangle$ be a field such that $b_0|\Phi\rangle \neq 0$. Since $|\Phi\rangle$ cannot be of level one, $L_0|\Phi\rangle \neq 0$. Then consider the following gauge equivalent state

$$|\widetilde{\Phi}\rangle = |\Phi\rangle - Q\frac{b_0}{L_0}|\Phi\rangle\,. \tag{241}$$

Using $\{b_0, Q\} = L_0$ one readily checks that $b_0|\widetilde{\Phi}\rangle = 0$, so the gauge can be reached. Moreover, we now show that no gauge transformation remains in this gauge. If there were, there would exist a non-zero string field in the gauge slice that happens to be pure gauge. Such field $|\Phi\rangle$ would then satisfy $b_0|\Phi\rangle = 0$, $L_0|\Phi\rangle \neq 0$, and $|\Phi\rangle = Q|\epsilon\rangle$. Since both b_0 and Q annihilate the state:

$$0 = b_0Q|\Phi\rangle + Qb_0|\Phi\rangle = \{b_0, Q\}|\Phi\rangle = L_0|\Phi\rangle\,, \tag{242}$$

in contradiction with the fact that the state does have non-zero dimension. The Siegel gauge is clearly a good gauge at the linearized level and within the twist truncated $\mathcal{H}_1$.

Let's now consider briefly the additional truncation that is allowed by $SU(1,1)$ symmetry (item (4) of our list). Once we work in the Siegel gauge, this further truncation is allowed. This truncation is only possible because of the particular form of the string vertex. It would not be allowed for arbitrarily defined star products. Let us recall how this symmetry arises in the cubic open string field theory [83]. In the Siegel gauge, the string field action reads

$$S \sim \frac{1}{2}\langle\phi|L_0|\phi\rangle + \frac{1}{3}\langle\phi|\langle\phi|\langle\phi\,|\,v_3\rangle\,. \tag{243}$$

The vertex coupling the three string fields is of the form

$$|v_3\rangle \sim \exp\left(E_{matt}\right)\;\exp\Big(-\sum_{r,s=1}^{3}\sum_{n,m=1}^{\infty} c^r_{-n}\,X^{rs}_{nm}\,b^s_{-m}\Big)\,|0_1\rangle_{123}\,, \tag{244}$$

where we have focused on the ghost sector. The Neumann coefficients are known to satisfy [83]

$$X^{rs}_{nm} = \frac{n}{m} X^{sr}_{mn}\,, \quad n, m \geq 1\,. \tag{245}$$

This relation is not true for general three string vertices, but holds for the open string field theory vertex. Given equation (245), the argument of the exponential in $|v_3\rangle_{123}$ can be written as a sum of terms of the form (r, s, n, m, not summed)

$$X^{rs}_{nm}\, c^r_{-n} b^s_{-m} + X^{sr}_{mn}\, c^s_{-m} b^r_{-n} = \frac{1}{m} X^{sr}_{mn} \Big(n c^r_{-n} b^s_{-m} + m c^s_{-m} b^r_{-n} \Big)\,. \tag{246}$$

The term in parenthesis is invariant under the continuous transformations

$$\begin{aligned} b_{-n}(\theta) &= b_{-n} \cos\theta - n c_{-n} \sin\theta\,, \\ c_{-n}(\theta) &= c_{-n} \cos\theta + \frac{1}{n} b_{-n} \sin\theta\,. \end{aligned} \tag{247}$$

These transformations, valid for all $n \neq 0$, imply $\{c_n(\theta), b_m(\theta)\} = \delta_{n+m}$. One readily finds that they are generated by an operator $\mathcal{S}_1$:

$$b_{-n}(\theta) = e^{\theta \mathcal{S}_1}\, b_{-n} e^{-\theta \mathcal{S}_1}\,, \quad c_{-n}(\theta) = e^{\theta \mathcal{S}_1}\, c_{-n} e^{-\theta \mathcal{S}_1}\,, \tag{248}$$

where $\mathcal{S}_1$ is given by

$$\mathcal{S}_1 = \sum_{n=1}^{\infty} \Big(\frac{1}{n}\, b_{-n} b_n - n c_{-n} c_n \Big)\,. \tag{249}$$

Since the vacuum $|0_1\rangle$ is annihilated by $\mathcal{S}_1$, the vertex $|v_3\rangle$ is invariant under this $U(1)$ symmetry: $\exp\Big(\theta(\mathcal{S}^{(1)}_1 + \mathcal{S}^{(2)}_1 + \mathcal{S}^{(3)}_1)\Big)|v_3\rangle_{123} = |v_3\rangle_{123}$. Equivalently,

$$\Big(\mathcal{S}^{(1)}_1 + \mathcal{S}^{(2)}_1 + \mathcal{S}^{(3)}_1\Big)|v_3\rangle_{123} = 0\,. \tag{250}$$

Since the vertex $|v_3\rangle_{123}$ is built from ghost bilinears of zero ghost number, we deduce that the ghost number operator $\mathcal{G}$

$$\mathcal{G} = \sum_{n=1}^{\infty} \Big(c_{-n} b_n - b_{-n} c_n \Big). \tag{251}$$

is also conserved:

$$\Big(\mathcal{G}^{(1)} + \mathcal{G}^{(2)} + \mathcal{G}^{(3)}\Big)|v_3\rangle_{123} = 0\,. \tag{252}$$

We can then form the commutator

$$[\mathcal{S}_1\,, \mathcal{G}\,] = 2\mathcal{S}_2\,, \quad \text{with} \quad \mathcal{S}_2 = \sum_{n=1}^{\infty} \Big(\frac{1}{n}\, b_{-n} b_n + n c_{-n} c_n \Big)\,. \tag{253}$$

The remaining commutators are readily computed:

$$[\mathcal{S}_2\,, \mathcal{G}\,] = 2\mathcal{S}_1\,, \quad [\mathcal{S}_1\,, \mathcal{S}_2\,] = -2\mathcal{G}\,. \tag{254}$$

These relations show that $\{\mathcal{S}_1, \mathcal{S}_2, \mathcal{G}\}$ generate the algebra of $SU(1,1)$. These generators are the same as those in the $SU(1,1)$ algebra in Siegel and Zwiebach [24].[4] Since both $\mathcal{S}_1$ and $\mathcal{G}$ are symmetries of the three string vertex, we also have

$$\left(\mathcal{S}_2^{(1)} + \mathcal{S}_2^{(2)} + \mathcal{S}_2^{(3)}\right)|v_3\rangle_{123} = 0\,. \tag{255}$$

In summary, the three string vertex is fully $SU(1,1)$ invariant.

The set of Fock space states built with the action of ghost and antighost oscillators on the vacuum $|0_1\rangle$ can be decomposed into finite dimensional irreducible representations of $SU(1,1)$. Note that (nc_{-n}, b_{-n}) transforms as a doublet. As usual, from the tensor product of two doublets one can obtain a nontrivial singlet; this is just

$$m b_{-n} c_{-m} + n b_{-m} c_{-n}\,. \tag{256}$$

It is now simple to argue that the twist even subspace of $\mathcal{H}_1$ in the Siegel gauge can be further restricted to $SU(1,1)$ singlets. Since the kinetic operator L_0 commutes with the $SU(1,1)$ generators, the kinetic term cannot couple a non-singlet to a singlet. Indeed, consider such a term $\langle s|L_0|a\rangle$, where $\langle s|$ is a singlet and $|a\rangle$ is not a singlet. Given the structure of the representations (completely analogous to the finite dimensional unitary representations of $SU(2)$), it follows that there is a state $|b\rangle$ and an $SU(1,1)$ generator $\mathcal{J}$ such that $|a\rangle = \mathcal{J}|b\rangle$. Therefore $\langle s|L_0|a\rangle = \langle s|L_0\mathcal{J}|b\rangle = \langle s|\mathcal{J}L_0|b\rangle = 0$, where the last step gives zero because $\mathcal{J}$ annihilates the singlet (this requires $bpz(\mathcal{J}) = \pm\mathcal{J}$, which is true). It remains to show that the vertex cannot couple a non-singlet to two singlets. Indeed, with analogous notation we have

$$\begin{aligned} {}_1\langle s_1|_2\langle s_2|_3\langle a\,|v_3\rangle \quad &= {}_1\langle s_1|_2\langle s_2|_3\langle b\,|\mathcal{J}^{(3)}|v_3\rangle \\ &= -{}_1\langle s_1|_2\langle s_2|_3\langle b\,|(\mathcal{J}^{(1)} + \mathcal{J}^{(2)})|\,v_3\rangle = 0\,, \end{aligned} \tag{257}$$

where we used the conservation of $\mathcal{J}$ on the vertex, and on the last step the $\mathcal{J}$ operators annihilate the singlets.

This completes our discussion of the various symmetries and conditions that can be used to constrain the subspace of the string state space that acquires vacuum expectation values in the tachyon vacuum.

7.3 The nonperturbative vacuum

Sen's first conjecture states that the string field theory action should lead to a nontrivial vacuum solution, with energy density

$$-T_{25} = -\frac{1}{2\pi^2 g^2}\,. \tag{258}$$

[4] Defining $X = (\mathcal{S}_2 - \mathcal{S}_1)/2$, $Y = (\mathcal{S}_2 + \mathcal{S}_1)/2$, and $H = \mathcal{G}$ we recover the conventional definition of the isomorphic (real) Lie algebra $sl(2,R)$, with brackets $[X,Y] = H$, $[H,X] = 2X$, $[H,Y] = -2Y$. Note that $T_+ = -2X$, where T_+ is the operator that multiplies b_0 in the BRST operator.

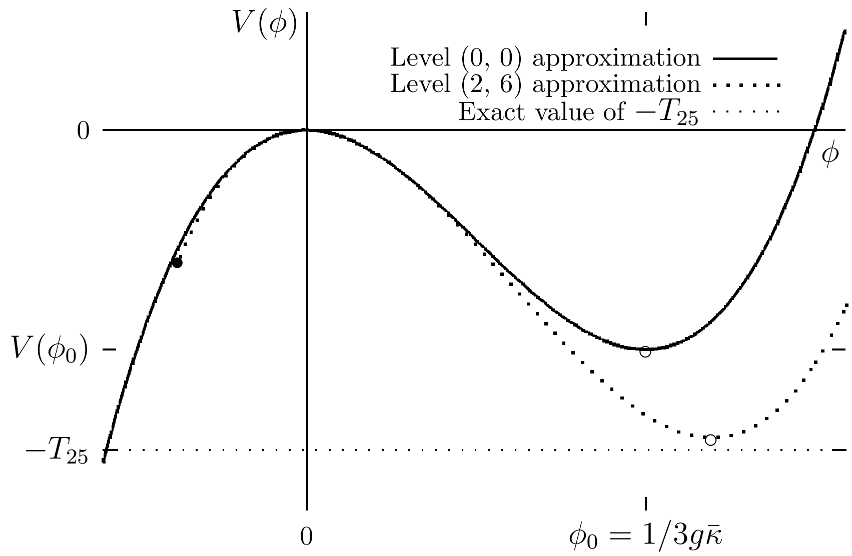


Figure 7: The effective tachyon potential in level (0, 0) and (2, 6) truncations. The open circles denote minima in each level truncation. The filled circle denotes a branch point where the level (2, 6) truncation approximation reaches the limit of Feynman-Siegel gauge validity.

In this subsection we discuss evidence for the validity of this conjecture in Witten's OSFT. As mentioned in the introduction, this result holds exactly in BSFT.

The string field theory equation of motion is

$$Q\Psi + g\Psi \star \Psi = 0\,. \tag{259}$$

Despite much work over the last few years, there is still no analytic solution of this equation of motion[5]. There is, however, a systematic approximation scheme, known as level truncation, which can be used to solve this equation numerically [57]. The level (L, I) truncation of the full string field theory involves dropping all fields at level $N > L$, and disregarding any interaction term between three fields whose levels add up to a number that is greater than I. For example, the simplest truncation of the theory is the level (0, 0) truncation. This is the truncation which was used in section 5.3. Including only the zero-momentum component of the tachyon field, since we are looking for a Lorentz-invariant vacuum, the truncated theory is simply described by a potential for the tachyon zero-mode

$$V(\phi) = -\frac{1}{2}\phi^2 + g\bar{\kappa}\phi^3\,. \tag{260}$$

where $\bar{\kappa} = \kappa/3 = 3^{7/2}/2^6$. This cubic function, which was computed in (123) using the CFT approach, and in (202) using the oscillator approach, is graphed in Figure 7. As discussed in section 5.3, this potential has a local minimum at

$$\phi_0 = \frac{1}{3g\bar{\kappa}}\,, \tag{261}$$

[5]as of October, 2003

and at this point the potential is

$$V(\phi_0) = -\frac{1}{54}\frac{1}{g^2\bar{\kappa}^2} = -\frac{2^{11}}{3^{10}}\frac{1}{g^2} \approx (0.68)\left(-\frac{1}{2\pi^2 g^2}\right). \tag{262}$$

Thus, simply including the tachyon zero-mode gives a nontrivial vacuum with 68% of the vacuum energy density predicted by Sen. This vacuum is denoted by an open circle in Figure 7.

At higher levels of truncation, there are a multitude of fields with various tensor structures. However, again assuming that we are looking for a vacuum which preserves Lorentz symmetry, we can restrict attention to the interactions between scalar fields at zero momentum. We will work in Feynman-Siegel gauge to simplify calculations; as shown in the previous subsection, this gauge is good at least in a local neighborhood of the point where all fields vanish. The situation is further simplified by the existence of the twist symmetry, which as mentioned in the previous subsection guarantees that no cubic vertex between (zero-momentum) scalar fields can connect three fields with a total level which is odd, and thus means that odd fields are not relevant to diagrams with only external tachyons at tree level. Therefore, we need only consider even-level scalar fields in looking for Lorentz-preserving solutions to the SFT equations of motion. With these simplifications, in a general level truncation the string field is simply expressed as a sum of a finite number of terms

$$\Psi_{\rm s} = \sum_i \phi_i |s_i\rangle \tag{263}$$

where ϕ_i are the zero-modes of the scalar fields associated with even-level states $|s_i\rangle$. As discussed in the previous subsection, this set of scalar fields can be further restricted to be SU(1, 1) singlets in the universal subspace $\mathcal{H}_1$. For example, including fields up to level 2, we have

$$\Psi_{\rm s} = \phi|0_1\rangle + B\,(\alpha_{-1}\cdot\alpha_{-1})|0_1\rangle + \beta\, b_{-1}c_{-1}|0_1\rangle\,. \tag{264}$$

In terms of the matter Virasoro generators, the state associated with the field B is

$$(\alpha_{-1}\cdot\alpha_{-1})|0_1\rangle = 2L_{-2}|0_1\rangle\,, \tag{265}$$

which lies in the universal subspace $\mathcal{H}_1$. The potential for all the scalars appearing in the level-truncated expansion (263) can be simply expressed as a cubic polynomial in the zero-modes of the scalar fields

$$V = \sum_{i,j} d_{ij}\phi_i\phi_j + g\bar{\kappa}\sum_{i,j,k} t_{ijk}\phi_i\phi_j\phi_k\,. \tag{266}$$

Using the expressions for the Neumann coefficients given in Section 5.3, the potential for all the scalar fields up to level L can be computed in a level (L, I) truncation. For example, the potential in the level $(2,6)$ truncation is given by

$$
\begin{aligned}
V &= -\frac{1}{2}\phi^2 + 26B^2 - \frac{1}{2}\beta^2 \qquad (267)\\
&+\bar{\kappa}g\left[\phi^3 - \frac{130}{9}\phi^2 B - \frac{11}{9}\phi^2\beta + \frac{30212}{243}\phi B^2 + \frac{2860}{243}\phi B\beta + \frac{19}{81}\phi\beta^2\right.\\
&\left.-\frac{2178904}{6561}B^3 - \frac{332332}{6561}B^2\beta - \frac{2470}{2187}B\beta^2 - \frac{1}{81}\beta^3\right]
\end{aligned}
$$

As an example of how these terms arise, consider the $\phi^2 B$ term. The coefficient in this term is given by

$$g\,\langle V_3|\left(|0_1\rangle \otimes |0_1\rangle \otimes \alpha_{-1}\cdot\alpha_{-1}|0_1\rangle\right) = -g\bar{\kappa}\,(3\cdot 26)\,V_{11}^{11} = -g\bar{\kappa}\frac{130}{9}\,, \qquad (268)$$

where we have used $V_{11}^{11} = 5/27$.

In the level (2, 6) truncation of the theory, the nontrivial vacuum is found by simultaneously solving the three quadratic equations found by setting to zero the derivatives of the potential (267) with respect to ϕ, B, and β. There are a number of different solutions to these equations, but only one is in the vicinity of $\phi = 1/3g\bar{\kappa}$. The solution of interest is

$$\phi \approx 0.39766\,\frac{1}{g\bar{\kappa}}\,, \quad B \approx 0.02045\,\frac{1}{g\bar{\kappa}}\,, \quad \beta \approx -0.13897\,\frac{1}{g\bar{\kappa}}\,. \qquad (269)$$

Plugging these values into the potential gives

$$E_{(2,6)} = -0.95938\,T_{25}\,, \qquad (270)$$

or 95.9% of the result predicted by Sen. This vacuum is denoted by an open circle in Figure 7.

It is a straightforward, computationally intensive project to generalize this calculation to higher levels of truncation. This calculation was carried out to level (4, 8) by Kostelecky and Samuel [57] many years ago. They noted that the vacuum seemed to be converging, but they lacked any physical picture to give meaning to this vacuum. Following Sen's conjectures, the level (4, 8) calculation was done again using somewhat different methods by Sen and Zwiebach [38], who showed that the energy at this level is $-0.986\,T_{25}$. The calculation was automated by Moeller and Taylor [84], who calculated up to level (10, 20), where there are 252 scalar fields, including all even-level scalar fields up to level 10; this computation was done using oscillators, without restriction to the universal subspace. Up to this level, the vacuum energy converges monotonically, as shown in Table 1. These numerical calculations indicate that level truncation of string field theory leads to a good systematic approximation scheme for computing the nonperturbative tachyon vacuum. It is also worth noting that in these computations, level $(L, 2L)$ and $(L, 3L)$ approximations give fairly similar values.

The preceding results were the best values for the vacuum energy at the time of these original lectures. More recently, Gaiotto and Rastelli reported further numerical results [85, 86]. By programming in C++ instead of mathematica, and by computing using matter Virasoro operators rather than oscillators, so that only fields in the universal subspace $\mathcal{H}_1$

level	$g\bar{\kappa}\langle\phi\rangle$	V/T_{25}
(0, 0)	0.3333	-0.68462
(2, 4)	0.3957	-0.94855
(2, 6)	0.3977	-0.95938
(4, 8)	0.4005	-0.98640
(4, 12)	0.4007	-0.98782
(6, 12)	0.4004	-0.99514
(6, 18)	0.4004	-0.99518
(8, 16)	0.3999	-0.99777
(8, 20)	0.3997	-0.99793
(10, 20)	0.3992	-0.99912

Table 1: Tachyon VEV and vacuum energy in stable vacua of level-truncated theory

level	V/T_{25}
(12, 24)	0.99979
(12, 36)	0.99982
(14, 28)	1.00016
(14, 42)	1.00017
(16, 32)	1.00037
(16, 48)	1.00038
(18, 36)	1.00049
(18, 54)	1.00049

Table 2: Vacuum energy in stable vacua of level-truncated theory

were included, they were able to extend the computation to level (18, 54). Their results are shown in Table 2 These results were rather surprising, indicating that while the energy monotonically approaches $-T_{25}$ up to level 12, at level (14, 42) the energy drops below $-T_{25}$, and that the energy continues to decrease, reaching $-1.00049\,T_{25}$ at level (18, 54). We will discuss the resolution of this unexpected overshoot shortly.

First, however, it is interesting to consider the tachyon condensation problem from the point of view of the effective tachyon potential. If instead of trying to solve the quadratic equations for all N of the fields appearing in (266), we instead fix the tachyon field ϕ and solve the quadratic equations for the remaining $N-1$ fields, we can determine a effective potential $V(\phi)$ for the tachyon field. This has been done numerically up to level (16, 48) [84, 86]. At each level, the tachyon effective potential smoothly interpolates between the perturbative vacuum and the nonperturbative vacuum near $\phi = 0.4/g\bar{\kappa}$. For example, the tachyon effective potential at level (2, 6) is graphed in Figure 7. In all level truncations other than (0, 0) and (2, 4) (at least up to level (10, 20)), the tachyon effective potential has two branch point singularities at which the continuous solution for the other fields breaks down; for the level (2, 6) truncation, these branch points occur at $\phi \approx -0.127/g\bar{\kappa}$ and $\phi \approx 2.293/g\bar{\kappa}$; the lower branch point is denoted by a solid circle in Figure 7. As a result of these branch points, the tachyon effective potential is only valid for a finite range of ϕ, ranging between approximately $-0.1/g\bar{\kappa}$ and $0.6/g\bar{\kappa}$. In Section 7.4 we review results which indicate that these branch points arise because the trajectory in field space associated with this potential encounters the boundary of the region of Feynman-Siegel gauge validity. It seems almost to be a fortunate accident that the nonperturbative vacuum lies within the region of validity of this gauge choice. It is worth mentioning again here that in the BSFT approach, the tachyon potential can be computed exactly [10]. In this formulation, there is no branch point in the effective potential, which is unbounded below for negative values of the tachyon. On the other hand, the nontrivial vacuum in the background-independent approach arises only as the tachyon field goes to infinity, so it is harder to study the physics of the stable vacuum from this point of view.

Another interesting perspective on the tachyon effective potential is found by performing a perturbative (but off-shell) computation of the coefficients in the tachyon effective potential in the level-truncated theory. This gives a power series expansion of the effective potential

$$\begin{aligned} V(\phi) &= \sum_{n=2}^{\infty} c_n (\bar{\kappa} g)^{n-2} \phi^n \\ &= -\frac{1}{2}\phi^2 + (\bar{\kappa} g)\phi^3 + c_4 (\bar{\kappa} g)^2 \phi^4 + c_5 (\bar{\kappa} g)^3 \phi^5 + \cdots \end{aligned} \tag{271}$$

The coefficients up to c_{60} have been computed in the level truncations up to (10, 20) [84]. Because of the branch point singularity near $\phi = -0.1/g\bar{\kappa}$, this series has a radius of convergence much smaller than the value of ϕ at the nonperturbative vacuum. Thus, the energy at the stable vacuum lies outside the naive range of the potential defined by the perturbative expansion.

Now, let us return to the problem of the overshoot in energy below $-T_{25}$ found at level 14 by Gaiotto and Rastelli[6]. The most straightforward way of determining whether or not this represents a real problem for string field theory would be to simply continue the calculation to higher levels. Unfortunately, at present this is not tractable, as the difficulty of computation grows exponentially in the level. Thus, we must resort to more indirect methods. It was found empirically by Taylor [87] that the level L approximations of string field theory give on-shell and off-shell amplitudes with error of order $1/L$. This work and further evidence [82, 88] indicates that amplitudes can be very accurately approximated by computing them in different level L truncations, and matching to a power series in $1/L$. Such an approach can be taken to determine highly accurate values for the coefficients c_n in (271). As noted above, the resulting power series has a finite radius of convergence, and the stable vacuum lies beyond this limit. There is a standard technique, however, known as the method of Padé approximants, which allows one to extrapolate a function beyond its naive radius of convergence, if the function is sufficiently well-behaved in the direction in which it is extrapolated. The idea of Padé approximants is to replace a power series having given coefficients for a fixed number of terms with a rational function with the same number of coefficients, choosing the coefficients of the rational function to give a power series which agrees with the fixed coefficients in the original power series. For example, consider the first three terms in the level $L = 2$ approximation to the tachyon effective potential (271),

$$-\frac{1}{2}\phi^2 + \kappa g \phi^3 - \frac{34}{27}(\kappa g)\phi^4 \,. \tag{272}$$

This truncated expansion has no local minima, while the Padé approximant

$$P_1^3(\phi) = \frac{-\frac{1}{2}\phi^2 + \frac{10}{27}\kappa g \phi^3}{1 + \frac{34}{27}\kappa g \phi} \tag{273}$$

does; this approximant thus represents a better description of the tachyon potential than the truncated expression (272). The advantage of Padé approximants is that they allow one to incorporate poles into approximations of a function with a desired local power series behavior. For a wide class of functions, successive Padé approximants converge exponentially quickly in the region where the function is smooth. Empirically, this seems to be the case for the tachyon effective potential. Thus, the energy minimum at any finite level of truncation can be determined to an arbitrary degree of accuracy from the leading coefficients in the potential. For example, the energy can be computed to 10 digits of accuracy by including approximately 40 coefficients c_n; this calculation is, however, highly sensitive to the accuracy of the coefficients [89].

Combining Padé approximants with approximations to the coefficients c_n, computed by matching level-truncated results in a $1/L$ expansion, it is possible to predict not only the exact value of the energy at the stable minimum as $L \to \infty$, but also to predict the values of the approximate energy at intermediate values of L. Such a computation was performed

[6] The material in the remainder of this section was developed only after the original TASI lectures in 2001, but is included because of its relevance to the main development in this section.

using the level approximated values of c_n up to level $(10, 20)$ [90]. By first using these values to predict the level-approximated values at higher levels, and then inserting these values into Padé approximants, the overshoot phenomenon found by Gaiotto and Rastelli was accurately reproduced. For example, compared to the value -1.0003678 found by these authors at level (16, 32), extrapolation from results at levels $(L, 2L)$ up to (10, 20) gives a predicted value of -1.0003773 at level (16, 32). Furthermore, the extrapolated values E_L of the energy at the stable minimum were found to decrease up to approximately level 26, and then to increase, approaching an asymptotic value as $L \to \infty$ of $E_\infty \approx -1$ with error $\sim 10^{-4}$. These results suggested that the energy at the minimum in the level-truncated theory takes the form shown in Figure 8.

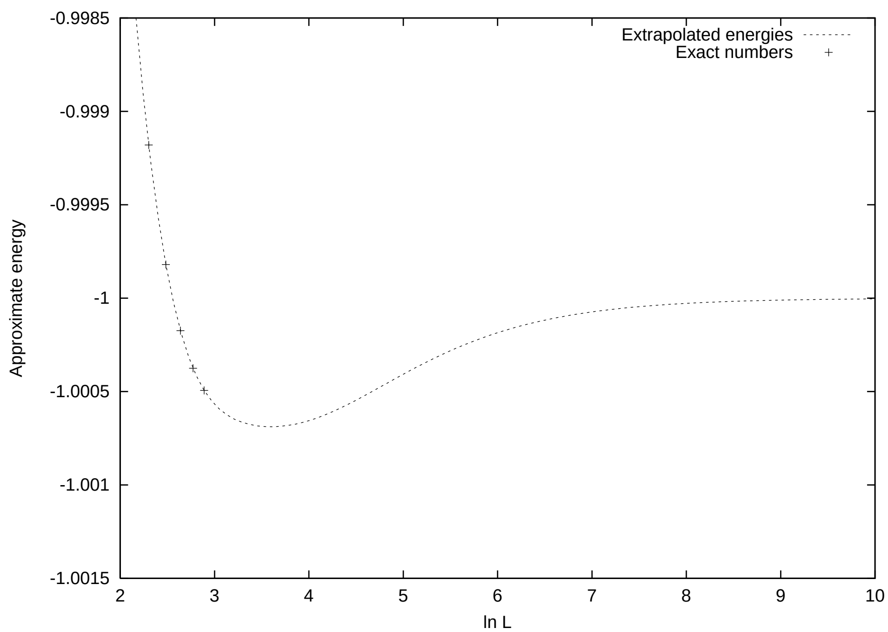


Figure 8: Expected approximations to the vacuum energy at different levels of truncation, extrapolated from data at lower levels of truncation.

In the calculation just described, there were two sources of error: 1) the coefficients c_n had some numerical inaccuracy, and 2) there is some error introduced in extrapolating from low levels of truncation.

This computation was improved by Gaiotto and Rastelli [86]. These authors used a different approach: instead of extrapolating the finite L results for the coefficients c_n, they extrapolated the nonperturbatively computed effective potential $V(\phi)$ at various values of ϕ. Because Padé approximants are so accurate, for exactly known values of c_n and $V(\phi)$ this approach is equivalent to the combined Padé-extrapolation in c_n approach, but generally this approach trades inaccuracy in c_n for inaccuracy in $V(\phi)$. In practice, it is much easier to compute the coefficients c_n exactly than the nonperturbative effective potential $V(\phi)$, which requires numerically solving a large system of quadratic equations. Gaiotto and Rastelli were able, however, to use their results on $V(\phi)$ at higher levels, which greatly increased the accuracy of their extrapolations. They found that while level $(L, 2L)$ and $(L, 3L)$ approximations tend to be very similar, extrapolations based on level $(L, 3L)$ truncations seem more robust. Using data up to level (16, 48) they found an extrapolated value of $E_\infty \approx -1.00003$, differing from -1 by an order of magnitude less than the value of the energy estimated at level 28, where the overshoot is predicted to be maximal. This gives compelling support to the conclusion that the level-truncated approximations to the energy indeed behave as shown in Figure 8, and approach the value predicted by Sen as $L \to \infty$.

7.4 Gauge fixing

In this subsection we discuss some aspects of the Feynman-Siegel gauge choice used in most explicit calculations in OSFT to date. Let us restrict attention to the zero momentum action for even-level scalar fields. This action is invariant under (54) with a general gauge parameter of the form

$$\Lambda = \sum \mu^a |s_a\rangle = \mu_1 b_{-2}|0_1\rangle + \cdots . \tag{274}$$

The ghost number zero states $|s_a\rangle$ are annihilated by b_0, so they do not contain c_0. The variation of a general zero-momentum scalar field takes the form

$$\delta\phi_i = D^{ia}\mu_a + g\bar{\kappa}\, T^{ija}\phi_j\mu_a\, . \tag{275}$$

At $\phi_i = 0$, we have the linear variation $\delta\phi_i = D^{ia}\mu_a$. Let ϕ_q denote fields associated with ghost number one states that contain a c_0. For example, at level two there is a field η associated with the state $c_0 b_{-2}|0_1\rangle$. At each level, the number of fields ϕ_q is clearly equal to the number of gauge parameters μ_a; the corresponding states are simply related by removing or replacing the c_0. From the formula for $Q_B = c_0 L_0 + \cdots$, it is easy to verify that D^{qa} is a linear one-to-one map at each level, so

$$\det D^{qa} \neq 0 \tag{276}$$

holds at each level. This is why the Feynman-Siegel gauge, which sets $\phi_q = 0$ at each level (and which limits us to gauge parameters associated with states without a c_0), is a good gauge choice near $\phi_i = 0$, as shown in subsection 7.2.

Let us now consider the gauge transformations at a general point in field space $\langle\phi_i\rangle$. We have

$$\delta\phi^i = M^{ia}\mu_a \tag{277}$$

where

$$M^{ia} = D^{ia} + g\bar{\kappa} T^{ija}\langle\phi_j\rangle\, . \tag{278}$$

Feynman-Siegel gauge breaks down whenever the determinant of this matrix vanishes

$$\det M^{qa} = 0\, . \tag{279}$$

This condition defines a region in field space within which Feynman-Siegel gauge is valid. At the boundary of this region, some gauge transformations give field variations which are tangent to the Feynman-Siegel gauge-fixed hypersurface. Some gauge orbits which cross the Feynman-Siegel gauge surface inside this region will cross again outside the region, giving a form of Gribov ambiguity. Furthermore, some gauge orbits never encounter the region of gauge validity. Thus, Feynman-Siegel gauge is really only locally valid.

We can study the region of Feynman-Siegel gauge validity in level truncation, using finite matrices M^{qa}. It is instructive to consider a simple example of the breakdown of this gauge choice. Consider dropping all fields other than the tachyon $\phi = \phi_1$ and the field $\eta = \phi_4$. The gauge transformation rules then become

$$\begin{aligned} \delta\phi &= \mu g\bar{\kappa}\left[-\frac{16}{9}\phi + \frac{128}{81}\eta\right] \\ \delta\eta &= -\mu + \mu g\bar{\kappa}\left[-\frac{224}{81}\phi + \frac{1792}{729}\eta\right]. \end{aligned} \tag{280}$$

In this simple model, M is a one-by-one matrix,

$$M = -\mu(1 + g\bar{\kappa}\frac{224}{81}\phi). \tag{281}$$

The gauge choice $\eta = 0$ breaks down when $\eta = \delta\eta = 0$ which occurs when

$$\phi = -\frac{1}{g\bar{\kappa}}\frac{81}{224}. \tag{282}$$

It is easy to see that smaller values of ϕ are gauge-equivalent to values of ϕ above this boundary value, while some gauge orbits never intersect the line $\eta = 0$.

The complete action including all even level (zero momentum) scalar fields and gauge invariances has been computed up to level (8, 16) [91]. One result of this computation is that the Feynman-Siegel gauge boundary condition $\det M^{qa} = 0$ seems to be very stable near the origin as the level of truncation is increased. This gives some confidence that there is a well-defined finite region in field space where Feynman-Siegel gauge is valid, and that the boundary of this region can be arbitrarily well approximated by level-truncation calculations. Another interesting result which can be seen from these calculations is that (to the precision possible in the level-truncated analysis) the branch points in the tachyon effective potential arise precisely at those points where the trajectory in field space associated with the effective potential crosses the Feynman-Siegel gauge boundary. Thus, these branch points are gauge artifacts. As mentioned previously, the tachyon effective potential computed from boundary string field theory does not suffer from such branch point problems.

It would be very desirable, however, to have an approach which enables one to describe the full string field space, including configurations which do not have gauge representatives in the local region of Feynman-Siegel gauge validity. Other gauge choices can be made, but those which have been explored to date are only minor variations on the Feynman-Siegel gauge choice, and do not lead to qualitatively different results. One might have hoped to isolate the true vacuum without gauge fixing at all, given that level truncation breaks the gauge symmetry and thus allows a discrete set of solutions at any level. This approach, however, is not particularly promising: the solutions found at each level lie at very different places on the gauge orbit, and do not approach any natural limit. Nonetheless, it seems of paramount importance to find some method for exploring the full field space of the theory. Currently inaccessible regions of the field space may contain solutions that have not yet been found (see subsection 7.6).

7.5 Lower-dimensional D-branes as solitons

One aspect of the Sen conjectures (item (2) in the list of section 3.3), proposes that lower-dimensional D-branes can be viewed as solitons of the D25-brane string field theory. The solitons involve profiles for the tachyon field which arise because the tachyon potential is non-trivial. The tachyon solitons are lumps, as opposed to kinks, which appear in superstring field theory solitons.

In this section we will discuss the basic ideas required to test this conjecture. We will follow the approach of Möeller, Sen, and Zwiebach [92] (other attempts [93] do not use level expansion). In order to be able to use a level expansion we curl up one spatial coordinate x into a circle of radius R (the corresponding string coordinate is called X). We will work with $R > 1$. Along this direction, we will wrap a D1-brane. We will then consider the possibility that a certain process of tachyon condensation results in the D1-brane becoming a D0-brane. Our use of D1- and D0-branes is just a matter of notational ease. Additional D-brane dimensions could be included.

Recall that the mass of the D1-brane can be written in the form

$$M_{D1} = 2\pi R T_1 = \frac{1}{2\pi^2 g^2} \tag{283}$$

where g is the coupling constant of the open string field theory that describes the D1-brane:

$$S = -\frac{1}{g^2}\Big(\frac{1}{2}\langle\Phi, Q\Phi\rangle + \frac{1}{3}\langle\Phi, \Phi * \Phi\rangle\Big) \equiv -\frac{1}{g^2}\mathcal{V}(\Phi)\,. \tag{284}$$

A few remarks are in order. In the above string action we have included into the string coupling factor $(1/g^2)$ the volume $(2\pi R)$ of the compact circle where the D1-brane is wrapped. By doing so, we can still use a CFT overlap with unit normalization, and the right-hand side in (283) gives the total mass of the brane. The zero string field here is supposed to describe the vacuum with a D1-brane stretched around the circle. For time-independent string fields (the kind of fields we consider here), $\mathcal{V}(\Phi)$ is a potential. More precisely the potential energy P.E. associated with a string field is

$$\text{P.E.} = -S(\Phi) = \frac{1}{g^2}\mathcal{V}(\Phi) = (2\pi R T_1)\, 2\pi^2\, \mathcal{V}(\Phi), \tag{285}$$

where we used (283). This potential energy is really the potential energy of field configurations measured with respect to the D1-brane background. Therefore, the total energy E_{tot} of the configuration is obtained by adding the energy of the D1-brane to the above P.E. We find

$$E_{tot}(\Phi) = (2\pi R T_1)\Big(1 + 2\pi^2\, \mathcal{V}(\Phi)\Big). \tag{286}$$

Since we will use the level expansion to investigate if a D0-brane can be represented as a lump solution, it is reasonable to use the level expansion to calculate the mass of the D1-brane, as well. So, we re-express the energy of the D1-brane in (286) in terms of the string field potential at the vacuum. Let $\Phi = T_{vac}$ denote the string field of the D1-brane SFT that represents the tachyon vacuum. Then, we have $-1 = 2\pi^2\mathcal{V}(T_{vac})$, and we can rewrite

$$E_{tot}(\Phi) = (2\pi RT_1)\Big(2\pi^2\,\mathcal{V}(\Phi) - 2\pi^2\,\mathcal{V}(T_{vac})\Big). \tag{287}$$

Indeed, this formula works correctly: when $\Phi = 0$ the total energy equals the mass of the D1-brane, and when $\Phi = T_{vac}$ the energy is zero (since the D1-brane has disappeared).

Let T_{lump} denote the lump (string field) solution, which is expected to represent the D0-brane in the field theory of the D1-brane. The energy of the lump solution is obtained from (287) for $\Phi = T_{lump}$:

$$E_{lump} = E_{tot}(T_{lump}) = (2\pi RT_1)\Big(2\pi^2\,\mathcal{V}(T_{lump}) - 2\pi^2\,\mathcal{V}(T_{vac})\Big). \tag{288}$$

The tensions T_0 and T_1 of the D0- and the D1-branes are related by $T_0 = 2\pi T_1$ (the D0-brane tension is the D0-brane energy). We can therefore form the ratio $r(R)$ of the lump energy and the D0-brane energy

$$r(R) = \frac{E_{lump}}{T_0} = R\Big(2\pi^2\,\mathcal{V}(T_{lump}) - 2\pi^2\,\mathcal{V}(T_{vac})\Big). \tag{289}$$

In the exact solution (or at infinite level), the ratio $r(R)$ should be equal to one. This is the content of the second tachyon conjecture. At any finite level $r(R)$ is some slowly varying function of R. Testing the conjecture for $R \to 1$ is quite difficult, and one must go to very high level in the computation. Testing the conjecture for R very large is also laborious, since many terms enter into any finite-level expansion. So, in practice, one chooses some reasonable value of R (the value $R = \sqrt{3}$ is convenient) and calculates to a fixed level.

Before reviewing some of the results obtained, let's do the simplest computation explicitly. We consider a tachyon field $T(x)$ which is expanded as

$$T(x) = t_0 + \sum_{n=1}^{\infty} t_n \cos(nx/R)\,. \tag{290}$$

The corresponding string field is written as

$$\begin{aligned} |T\rangle \quad &= t_0 c_1|0\rangle + \sum_{n=1}^{\infty}\frac{1}{2}t_n\Big(e^{inX(0)/R} + e^{-inX(0)/R}\Big)\,c_1|0\rangle\,,\\ &= t_0 c_1|0\rangle + \sum_{n=1}^{\infty}\frac{1}{2}\,t_n\Big(c_1|n/R\rangle + c_1|-n/R\rangle\Big)\,. \end{aligned} \tag{291}$$

We now evaluate the string action, keeping t_0 and the first tachyon harmonic t_1:

$$|T\rangle = t_0 c_1|0\rangle + \frac{1}{2}\,t_1\Big(c_1|1/R\rangle + c_1|-1/R\rangle\Big)\,. \tag{292}$$

Consider first the contribution of t_1 to the kinetic term

$$
\begin{aligned}
\frac{1}{2}\langle T, QT\rangle\Big|_{t_1} &= \frac{1}{2}\frac{t_1}{2}\cdot\frac{t_1}{2}\Big(\langle 1/R| + \langle -1/R|\Big)c_{-1}c_0L_0c_1\Big(|1/R\rangle + |-1/R\rangle\Big) \\
&= \frac{1}{4}t_1^2\Big(-1 + \frac{1}{R^2}\Big)\,.
\end{aligned} \tag{293}
$$

Note that, as mentioned earlier, the overlaps have unit normalization. Let us now calculate the terms that arise from the interaction. Because of momentum conservation there are no t_1^3 or $t_1t_0^2$ terms. There is only a $t_1^2t_0$ coupling, which is readily calculated as

$$
\frac{1}{3}\cdot 3\cdot\frac{t_1}{2}\cdot\frac{t_1}{2}t_0\cdot 2\cdot\langle c_1e^{iX/R}, ce^{-iX/R}, c\rangle = \frac{1}{2}t_0t_1^2K^{3-\frac{2}{R^2}}\,. \tag{294}
$$

Let us explain the origin of the various factors. The first 1/3 is the one that comes with the interaction term in the action. The factor of 3 is because there are three possible places to insert the operator associated with t_0. The factors of $t_1/2$ and t_0 come from the field expansion, and the factor of two arises because there are two ways in which the momentum can be conserved. The correlator has been evaluated in a way similar to the previous computation that led to (122). Indeed, the only difference is that the conformal dimension of two of the operators has been shifted from -1 to $-1+\frac{1}{R^2}$.

Collecting now our results and using the previously calculated potential for t_0 (123) we find

$$
\mathcal{V}(t_0, t_1) = -\frac{1}{2}t_0^2 - \frac{1}{4}\Big(1 - \frac{1}{R^2}\Big)t_1^2 + \frac{1}{3}K^3t_0^3 + \frac{1}{2}t_0t_1^2K^{3-\frac{2}{R^2}}\,. \tag{295}
$$

The original tachyon is still there: it corresponds to the field t_0, which in the present expansion has no momentum. For $R>1$, the field t_1 is also a tachyon. This field is present because of the instability to form a D1-brane. Indeed, for $R>1$ the energy of the D1-brane is larger than the energy of the D0-brane, and the decay is possible. For $R<1$, the D0-brane has more energy than the D1-brane. In this case, it is not clear if some high level computation can exhibit the D0-brane as a solution of the D1-brane field theory. We return to this problem in the next subsection.

Let's take $R=\sqrt{3}$. In this case the potential $\mathcal{V}(t_0,t_1)$ has a critical point which represents a lump: $t_0\simeq 0.18$ and $t_1=-0.34$. Of course there is also the conventional tachyon vacuum solution with $t_0 = 1/K^3$ and $t_1 = 0$. With these two solutions, one can readily compute the ratio $r(\sqrt{3})$ in (289). We find $r(\sqrt{3})\simeq 0.774$ in this lowest order calculation. The result is certainly quite good. This computation is called a level (1/3; 2/3) computation since the highest level field t_1 has level $1/3 = 1/R^2$, and we kept terms in the potential up to level 2/3. A computation at level (2,4) gives $r\simeq 1.02$, and for level (3,6) one finds $r\simeq 0.994$. The convergence to the answer is quite spectacular. This computation includes the tachyon harmonics t_1, t_2, and t_3, as well as fields from the second level and their first harmonics. No higher level computations have been done for this problem. The computations are not completely universal since the Virasoro structure of the state space depends on the radius of the circle. For rational values of R one may find null states, so this is why we took R irrational. Even for R irrational, not all states can be written as Virasoro descendents of the vacuum $|0\rangle$. New primaries (and their descendents) are needed starting at level 4.

Since we are equipped with the tachyon harmonics, one is able to construct explicitly the tachyon profile for the lump solution which represents the D0-brane. As the level is increased, the profile appears to settle into a well-defined limit. That same profile appears to arise for various values of the radius R of the circle used for the computation. The profile is roughly of the form

$$T(x) \simeq a + b\, e^{-x^2/(2\sigma^2)}\,, \quad a \simeq 0.56, \quad b \simeq -0.83, \quad \sigma \simeq 1.52\,. \tag{296}$$

The σ width of the lump is therefore about $1.5\sqrt{\alpha'}$. The significance (or gauge independence) of this width is not clear. Nevertheless, it is interesting that D-branes, which are defined by definite positions in CFT, appear as thick objects in SFT. Physical questions regarding D-branes are expected to have identical answers in the two approaches.

The above computations have been generalized to the case of lump solutions of codimension two. In this case, we can imagine a D2-brane wrapped on a torus T^2 which decays into a D0-brane. The results in the level expansion appear to confirm that the lump solutions do represent D0-branes. Less accurate results are obtained; the energy has only been estimated with about ten percent accuracy.

The above results have simple analogs in field theory [94]. Consider a simple scalar field theory in $p+1$ spatial dimensions, where we single out a coordinate x for special treatment:

$$S = \int dt d^p y dx \Big\{ \frac{1}{2}\Big(\frac{\partial\phi}{\partial t}\Big)^2 - \frac{1}{2}|\nabla_y\phi|^2 - \frac{1}{2}\Big(\frac{\partial\phi}{\partial x}\Big)^2 - V(\phi)\Big\}\,. \tag{297}$$

As you can readily verify, time-independent solitonic solutions $\phi(x)$, which depend only on the coordinate x, are obtained by solving the second-order ordinary differential equation

$$\frac{d^2\phi}{dx^2} = V'(\phi(x))\,. \tag{298}$$

This equation takes the form of the equation of motion of a unit mass particle in a one-dimensional potential $-V(x)$. As an example, we consider a theory with potential [95]

$$V(\phi) = \frac{1}{3}(\phi-1)^2\left(\phi + \frac{1}{2}\right)\,. \tag{299}$$

The potential has a maximum at $\phi = 0$ and a local minimum at $\phi = 1$. At $\phi = 0$ the interpretation is that of a D$(p+1)$-brane with tension

$$T_{p+1} = V(\phi = 0) = \frac{1}{6}\,. \tag{300}$$

As a simple exercise, verify that

$$\phi(x) = 1 - \frac{3}{2}\,\mathrm{sech}^2(x/2)\,, \tag{301}$$

is a lump solution for this potential.
Exercise: Show that the lump solution is an object with tension $T_p = 6/5$.

In string theory the ratio $\frac{1}{2\pi}\frac{T_p}{T_{p+1}}$ is equal to one. In this field theory model with a cubic potential, we find

$$\frac{1}{2\pi}\frac{T_p}{T_{p+1}} = \frac{1}{2\pi}\frac{6}{5}\cdot 6 = \frac{18}{5\pi} \simeq 1.146\,. \tag{302}$$

It is also a familiar result in soliton field theory that the spectrum of excitations that live on the world-volume of the lump solution $\bar{\phi}(x)$ is governed by a Schrödinger equation with a potential $V''(\bar{\phi}(x))$. The mass-squared values for the modes that live on the lump coincide with the Schrödinger energies.

There has been some interest in finding potentials that accurately describe the behavior of the tachyon. While the kinetic terms are not standard, the potential

$$V(\phi) = -\frac{1}{4}\phi^2 \ln \phi^2\,, \quad \phi > 0\,. \tag{303}$$

appears to be an exact effective tachyon potential. This potential was obtained [96] in an attempt to construct realistic tachyon potentials, and was later confirmed to appear in the BSFT approach to string field theory [10]. The tachyon vacuum is at $\phi = 0$, and surprisingly (but correctly!) the tachyon mass goes to infinity at this vacuum. This is consistent with the conjecture that perturbative open string degrees of freedom disappear at the tachyon vacuum.
Exercise. Show that $\bar{\phi}(x) = \exp(-x^2/4)$ is the lump solution for the potential (303) and the Schrödinger potential for fluctuations on the lump solution is $\frac{x^2}{4} - \frac{3}{2}$, a simple harmonic oscillator potential. Finally, confirm that the values of m^2 for the particles that live on the lump are $-1, 0, 1, 2, \ldots$. This is the expected string spectrum!

7.6 Open string theory backgrounds

We mentioned in the last subsection that when the radius R of a circle on which a D1-brane is compactified becomes small, it is not known how to represent a D0-brane in the string field theory on the D1-brane. When $R < 1$, the energy of the resulting D0-brane is larger than the energy of the original D1-brane. Thus, such a solution would have positive energy with respect to the original system. The difficulty of constructing such a D0-brane solution is an example of a more general, and we believe crucial, question for OSFT: Does OSFT, either through level truncation or some more sophisticated analytic approach, admit classical solutions which describe open string backgrounds with higher energy than the configuration with respect to which the theory is originally defined? If OSFT is to be a truly complete formulation of string theory, such solutions must be possible, since all open string backgrounds must be accessible to the theory.

Another problem of this type is to find, either analytically or numerically, a solution of the OSFT formulated with one D25-brane that describes *two* D25-branes. It should be just as feasible to go from a vacuum with one D-brane to a vacuum with two D-branes as it is to go from a vacuum with one D-brane to the empty vacuum. Despite some work on this problem [97], there is as yet no evidence of a solution. Several approaches which have been tried include: *i*) following a positive mass field upward, looking for a stable point; this method seems to fail because of gauge-fixing problems—the effective potential often develops a singularity before reaching the energy $+T_{25}$, *ii*) following the intuition of the RSZ model (discussed in the following section) and constructing a gauge transform of the original D-brane solution which is $\star-$orthogonal to the original D-brane vacuum. It can be shown formally that such a state, when added to the original D-brane vacuum gives a new solution with the correct energy for a double D-brane; unfortunately, however, we have been unable to identify such a state numerically in level truncation.

While so far no progress has been made towards the construction of solutions with higher energy than the initial vacuum, it is also interesting to consider the marginal case. An example of such a situation is embodied is the problem of translating a single D-brane of less than maximal dimension in a transverse direction. It was shown by Sen and Zwiebach [98] (in a T-dual picture) that after moving a D-brane a finite distance of order of the string length in a transverse direction, the level-truncated string field theory equations develop a singularity. Thus, in level truncation it does not seem possible to move a D-brane a macroscopic distance in a transverse direction[7]. In this case, a toy model [95] suggests that the problem is that the infinitesimal marginal parameter for the brane translation ceases to parameterize the marginal trajectory in field space after a finite distance, just as the coordinate x ceases to parameterize the circle $x^2 + y^2 = 1$ near $x = 1$. Indeed, an explicit calculation [82] of the field redefinition needed to take the OSFT field A associated with the transverse motion to the correct marginal parameter a shows that this field redefinition has a subleading term

$$A = a + \alpha a^3 + \cdots , \tag{304}$$

where $\alpha < 0$. Thus, as a increases, eventually a point is reached where A begins to decrease. This shows that A is not a good parameter for marginal deformations of arbitrary size. It would be nice to have a clear understanding of how arbitrary marginal deformations are encoded in the theory.

To show that open string field theory is sufficiently general to address arbitrary questions involving different vacua, it is clearly necessary to show that the formalism is powerful enough to describe multiple brane vacua, the D0-brane lump on an arbitrary radius circle, and translated brane vacua. It is currently unclear whether the obstacles to finding these vacua are technical or conceptual. It may be that the level-truncation approach is not well-suited to finding these vacua, and a new approach is needed.

8 String field theory around the stable vacuum

The tachyon conjectures state that the classically stable vacuum is the closed string vacuum. This implies that there should be no open string excitations in this vacuum, given that the D-brane represented by the original OSFT has decayed and exists no more. Without a D-brane

[7]Although this can be done formally [99], it is unclear how the formal solution relates to an explicit expression in the oscillator language.

conventional perturbative open string states are not expected to exist. If any perturbative states exist in this vacuum, they should be closed string states, which are only expected to appear in the quantum open string field theory.

There are two natural questions concerning this conjecture. First, we ask: Can it be tested? For this, we can begin with the original OSFT on the background of a D25-brane, for example, and use the (numerical) solution Φ_0 for the tachyon vacuum to expand the classical OSFT around the tachyon vacuum and to calculate the spectrum. The conjecture requires that no physical states be encountered. Second, we ask: Is there a more natural formulation of open string theory around the tachyon vacuum, in which, for example, the background independence of the theory might be more manifest? The theory around the tachyon vacuum, is, no doubt, rather unusual. In the tachyon vacuum there are no apparent physical states, at least none that take any familiar form. Physical perturbative states can arise only from quantum effects or classically after the theory is shifted to a nontrivial background that represents some D-brane configuration.

The tachyon vacuum is a rather special vacuum: it is the end product of the decay of *any* D-brane configuration. Presumably, the theory at the tachyon vacuum is independent of the particular version of OSFT used to reach it upon tachyon condensation, in the sense that the string field theories associated with different D-brane configurations should be equivalent under field redefinition around the stable vacuum of each theory. If that is the case, there may exist a theory – which we can call *Vacuum String Field Theory*, or VSFT – which formulates the physics of the tachyon vacuum directly, *without* using a D-brane background to reach the tachyon vacuum.

Presently, there is background dependence in the formulation of Witten's OSFT; some specific D-brane background must be chosen to define the theory, even though this D-brane configuration may be removed through tachyon condensation. As a result, even if the theory is in an abstract sense completely background independent, we are stuck with some particular choice of "coordinates" on the theory arising from the original choice of background, which may make physics in other backgrounds rather difficult to disentangle. The tachyon vacuum is also a specific background, but it is certainly a choice that is more canonical than one which picks one out of an infinite number of D-brane configurations. There are perhaps two canonical choices: an infinite number of space-filling D-branes, which has been motivated from the viewpoint of K-theory [17], and a background with no D-branes whatsoever – the tachyon vacuum. In this section we investigate the second choice.

A strikingly simple formulation of VSFT was proposed by Rastelli, Sen, and Zwiebach (RSZ) [100], in which the BRST operator is taken to be purely contained in the ghost sector. In this theory, closed-form analytic solutions that represent D-branes can be found and take the form of projectors of the star-algebra. One shortcoming of this VSFT is that certain computations are singular and require regularization. It remains to be seen if a regular VSFT exists.

In subsection 8.1 we describe the form of the OSFT action when expanded around the classically stable tachyon vacuum. Subsection 8.2 describes evidence from Witten's OSFT that the open string degrees of freedom truly disappear from the theory in this vacuum. In 8.3 we introduce and discuss the RSZ model of VSFT. Subsection 8.4 describes an important class of states in the star algebra: slivers and projectors, which play a key role in constructing D-branes in the RSZ model, and which may also be useful in understanding solutions of the Witten theory. Finally, in 8.5 we discuss closed strings in OSFT.

8.1 String field theory in the true vacuum

We have seen that numerical results from level-truncated string field theory strongly suggest the existence of a classically stable vacuum solution Φ_0 to the string field theory equation of motion. The state Φ_0, while still unknown analytically, has been determined numerically to a high degree of precision. This state seems like a very well-behaved string field configuration. While there is no positive-definite inner product on the string field Fock space, the state Φ_0 certainly has finite norm under the natural inner product $\langle V_2|\Phi_0, c_0 L_0 \Phi_0\rangle$, and is even better behaved under the product $\langle V_2|\Phi_0, c_0\Phi_0\rangle$. Thus, it is natural to assume that Φ_0 defines a classically stable vacuum for the theory, around which we can expand the action to find a string field theory around the tachyon vacuum.

Let Φ_0 be the string field configuration describing the tachyon vacuum. This string field satisfies the classical field equation

$$Q\Phi_0 + \Phi_0 * \Phi_0 = 0\,. \tag{305}$$

If $\widetilde{\Phi} = \Phi - \Phi_0$ denotes the shifted open string field, then the cubic string field theory action (61) expanded around the tachyon vacuum has the form:

$$S(\Phi_0 + \widetilde{\Phi}) = S(\Phi_0) \; - \; \frac{1}{g^2}\left[\frac{1}{2}\langle\,\widetilde{\Phi}\,,\,\widetilde{Q}\,\widetilde{\Phi}\rangle + \frac{1}{3}\langle\,\widetilde{\Phi}\,,\,\widetilde{\Phi} * \widetilde{\Phi}\rangle\right]. \tag{306}$$

Here $S(\Phi_0)$ is a constant, which according to the energetics part of the tachyon conjectures equals the tension of the D-brane times its volume (as before, we assume that the time interval has unit length so that the action can be identified with the negative of the potential energy for static configurations). The kinetic operator $\widetilde{Q}$ is given in terms of Q and Φ_0 as:

$$\widetilde{Q}\widetilde{\Phi} = Q\widetilde{\Phi} + \Phi_0 * \widetilde{\Phi} + \widetilde{\Phi} * \Phi_0\,. \tag{307}$$

More generally, on arbitrary string fields one would define

$$\widetilde{Q}A = QA + \Phi_0 * A - (-1)^A A * \Phi_0\,. \tag{308}$$

The consistency of the new action (306) is guaranteed from the consistency of the action in (61). Since neither the inner product nor the star multiplication have changed, the identities in (63) still hold. One can also check that the identities in (62) hold when Q is replaced by $\widetilde{Q}$. Just as the original action is invariant under the gauge transformations (71), the new action is invariant under $\delta\widetilde{\Phi} = \widetilde{Q}\Lambda + \widetilde{\Phi} * \Lambda - \Lambda * \widetilde{\Phi}$ for any Grassmann-even ghost-number zero state Λ.

Since the energy density of the brane represents a positive cosmological constant, it is natural to add the constant $-M = -S(\Phi_0)$ to (61). This will cancel the $S(\Phi_0)$ term in (306), and will make manifest the expected zero energy density in the final vacuum without D-brane. For the analysis around this final vacuum it suffices therefore to study the action

$$S_0(\widetilde{\Phi}) \equiv -\frac{1}{g^2}\left[\frac{1}{2}\langle \widetilde{\Phi}, \widetilde{Q}\,\widetilde{\Phi}\rangle + \frac{1}{3}\langle \widetilde{\Phi}, \widetilde{\Phi} * \widetilde{\Phi}\rangle\right]. \tag{309}$$

This string field theory around the stable vacuum has precisely the same form as Witten's original cubic string field theory, only with a different BRST operator $\widetilde{Q}$, which so far is only determined numerically. While this is insufficient for a complete formulation, it suffices to test the conjecture that open string excitations disappear in the tachyon vacuum, as we will discuss in Section 8.2

The numerical solution for Φ_0 provides a numerical definition of the string field theory around the tachyon vacuum. How could we do better? If we had a closed form solution Φ_0 available, the problem of formulating SFT around the tachyon vacuum would be significantly simplified. It is not clear, however, that the resulting formulation would be the best possible one. Previous experience with background deformations (small and large) in SFT indicates that even if we knew Φ_0 explicitly and constructed $S_0(\widetilde{\Phi})$ using eq.(309), this may not be the most convenient form of the action. Typically a nontrivial field redefinition is necessary to bring the shifted SFT action to the canonical form representing the new background [101]. In fact, in some cases, such as in the formulation of open SFT for D-branes with various values of magnetic fields, it is possible to formulate the various SFT's directly [102, 103], but the nontrivial classical solution relating theories with different magnetic fields are not known. This suggests that if a simple form exists for the SFT action around the tachyon vacuum it might be easier to guess it than to derive it.

In fact, this is exactly the approach to the formulation of vacuum string field theory (VSFT) taken by Rastelli, Sen, and Zwiebach (RSZ) [100]. These authors postulate that at the tachyon vacuum the action takes the form

$$\mathcal{S}(\Phi) \equiv -K_0\left[\frac{1}{2}\langle \Phi, \mathcal{Q}\,\Phi\rangle + \frac{1}{3}\langle \Phi, \Phi * \Phi\rangle\right], \tag{310}$$

where the new kinetic operator $\mathcal{Q}$ is an operator build solely out of ghosts fields. If this gives a consistent theory at the tachyon vacuum, they argue, their choice of $\mathcal{Q}$ must be field redefinition equivalent to the $\widetilde{Q}$ that arises directly by shifting the original OSFT action with the tachyon solution Φ_0. We discuss the RSZ model in section 8.3.

8.2 Decoupling of open strings

It may seem surprising to imagine that *all* the perturbative open string degrees of freedom will vanish at a particular point in field space, since these are all the apparent degrees of freedom available in the theory. This is not a familiar phenomenon from quantum field theory. To understand how the open strings can decouple, it may be helpful to begin by considering the simple example of the (0, 0) level-truncated theory. In this theory, the quadratic terms in the action become

$$-\int d^{26}p\ \phi(-p)\left[\frac{p^2-1}{2}+g\bar{\kappa}\left(\frac{16}{27}\right)^{p^2}\cdot 3\langle\phi\rangle\right]\phi(p)\,. \qquad (311)$$

Taking $\langle\phi\rangle = 1/3\bar{\kappa}g$, we find that the quadratic term is a transcendental expression which does not vanish for any real value of p^2. Thus, this theory has no poles, and the tachyon has decoupled from the theory. Of course, this is not the full story, as there are still finite complex poles. It does, however suggest a mechanism by which the nonlocal parts of the action (encoded in the exponential of p^2) can remove physical poles.

To get the full story, it is necessary to continue the analysis to higher level. At level 2, there are 7 scalar fields, the tachyon and the 6 fields associated with the Fock space states

$$\begin{array}{rl}(\alpha_{-1}\cdot\alpha_{-1})|0_1,p\rangle & b_{-1}\cdot c_{-1}|0_1,p\rangle \\ c_0\cdot b_{-1}|0_1,p\rangle & (p\cdot\alpha_{-2})|0_1,p\rangle \\ (p\cdot\alpha_{-1})^2|0_1,p\rangle & (p\cdot\alpha_{-1})c_0 b_1|0_1,p\rangle\end{array} \qquad (312)$$

Note that in this analysis we cannot fix Feynman-Siegel gauge, as we only believe that this gauge is valid for the zero-modes of the scalar fields in the vacuum Ψ_0. An attempt at analyzing the spectrum of the theory in Feynman-Siegel gauge using level truncation has been made [57], but gave no sensible results[8]. Diagonalizing the quadratic term in the action on the full set of 7 fields of level ≤ 2, we find [105] that poles develop at $M^2 = 0.9$ and $M^2 = 2.0$ (in string units, where the tachyon has $M^2 = -1$). These poles correspond to states satisfying $\tilde{Q}\tilde{\Psi} = 0$. The question now is, are these states physical? If they are exact states, of the form $\tilde{\Psi} = \tilde{Q}\tilde{\Lambda}$, then they are simply gauge degrees of freedom. If not, however, then they are states in the cohomology of $\tilde{Q}$ and should be associated with physical degrees of freedom. Unfortunately, we cannot precisely determine whether the poles we find in level truncation are due to exact states, as the level-truncation procedure breaks the condition $\tilde{Q}^2 = 0$. Thus, we can only measure *approximately* whether a state is exact. A detailed analysis of this question was carried out by Ellwood and Taylor [105]. In their paper, all terms in the SFT action of the form $\phi_i\ \psi_j(p)\ \psi_k(-p)$ were determined, where ϕ_i is a scalar zero-mode, and $\psi_{j,k}$ are nonzero-momentum scalars. In addition, all gauge transformations involving at least one zero-momentum field were computed up to level (6, 12). At each level up to $L = 6$, the ghost number 1 states in the kernel Ker $\tilde{Q}^{(1)}_{(L,2L)}$ were computed. The extent to which each of these states lies in the exact subspace was measured using the formula

$$\%\ \text{exactness} = \sum_i \frac{(s\cdot e_i)^2}{(s\cdot s)} \qquad (313)$$

[8]Note added: Recently, Giusto and Imbimbo have carried out a more detailed analysis of the spectrum around the stable vacuum in Feynman-Siegel gauge [104]. Their more careful analysis shows that spurious poles found in [57] are cancelled when the truncation level is sufficiently high. This approach gives a nice confirmation of the results of [105], while working in a fixed gauge, and extends these results by having sensitivity to cohomology associated with states which are closed for all momentum, but not exact at discrete values of momentum (type A states in the notation of [104]).

where $\{e_i\}$ are an orthonormal basis for Im $\tilde{Q}^{(0)}_{(L,2L)}$, the image of $\tilde{Q}$ acting on the space of ghost number 0 states in the appropriate level truncation. (Note that this measure involves a choice of inner product on the Fock space; several natural inner products were tried, giving roughly equivalent results). The result of this analysis was that up to the mass scale of the level truncation, $M^2 \leq L - 1$, all the states in the kernel of $\tilde{Q}^{(1)}$ were $\geq 99.9\%$ within the exact subspace, for $L \geq 4$. This result seems to give very strong evidence for Sen's third conjecture that there are no perturbative open string excitations around the stable classical vacuum Ψ_0. This analysis was only carried out for even level scalar fields; it would be nice to check that a similar result holds for odd-level fields and for tensor fields of arbitrary rank.

Another more abstract argument that there are no open string states in the stable vacuum was given by Ellwood, Feng, He and Moeller [63]. These authors argued that in the stable vacuum, the identity state $|I\rangle$ in the SFT star algebra, which satisfies $I \star A = A$ for a very general class of string fields A, seems to be an exact state,

$$|I\rangle \;\; = \;\; \tilde{Q}|\Lambda\rangle\,. \tag{314}$$

If indeed the identity is exact, then it follows immediately that the cohomology of $\tilde{Q}$ is empty, since $\tilde{Q}A = 0$ then implies that

$$A = (\tilde{Q}\Lambda) \star A = \tilde{Q}(\Lambda \star A) - \Lambda \star \tilde{Q}A = \tilde{Q}(\Lambda \star A)\,. \tag{315}$$

So to prove that the cohomology of $\tilde{Q}$ is trivial, it suffices to show that $\tilde{Q}|\Lambda\rangle = |I\rangle$. While there are some subtleties involved with the identity string field, Ellwood *et al.* found a very elegant expression for this field,

$$|I\rangle \;\; = \;\; \left(\cdots e^{\frac{1}{8}L_{-16}} e^{\frac{1}{4}L_{-8}} e^{\frac{1}{2}L_{-4}}\right) e^{L_{-2}}|0\rangle\,. \tag{316}$$

(Recall that $|0\rangle = b_{-1}|0_1\rangle$.) They then looked numerically for a state $|\Lambda\rangle$ satisfying (314). For example, truncating at level $L = 3$,

$$\begin{aligned} |I\rangle \;\; &= \;\; |0\rangle + L_{-2}|0\rangle + \cdots \\ &= \;\; |0\rangle - b_{-3}c_1|0\rangle - 2b_{-2}c_0|0\rangle + \frac{1}{2}(\alpha_{-1}\cdot\alpha_{-1})|0\rangle + \cdots \end{aligned} \tag{317}$$

while the only candidate for $|\Lambda\rangle$ is

$$|\Lambda\rangle = \alpha\; b_{-2}|0\rangle, \tag{318}$$

for some constant α. The authors of [63] showed that the state (317) is best approximated as exact when $\alpha \sim 1.12$; for this value, their measure of exactness becomes

$$\frac{\left|\tilde{Q}|\Lambda\rangle - |I\rangle\right|}{|I|} \to 0.17, \tag{319}$$

which the authors interpreted as a 17% deviation from exactness. Generalizing this analysis to higher levels, they found at levels 5, 7, and 9, a deviation from exactness of $11\%, 4.5\%$

and 3.5% respectively. At level 9, for example, the identity field has 118 components, and there are only 43 gauge parameters, so this is a highly nontrivial check on the exactness of the identity. Like the results of Ellwood and Taylor [105], these results strongly support the conclusion that the cohomology of the theory is trivial in the stable vacuum. In this case, the result applies to fields of all spins and all ghost numbers.

Given that the Witten string field theory seems to have a classical solution with no perturbative open string excitations, in accordance with Sen's conjectures, it is quite interesting to ask what the physics of the string field theory in the stable vacuum should describe. One natural assumption might be that this theory should include closed string states in its quantum spectrum. Unfortunately, addressing this question requires performing calculations in the quantum theory around the stable vacuum. Such calculations are quite difficult (although progress in this direction has been made by Minahan in the p-adic version of the theory [106]). Even in the perturbative vacuum, it is difficult to systematically study closed strings in the quantum string from theory. We discuss this question again in the final subsection of this section.

8.3 Pure ghost Vacuum String Field Theory

Our discussion in Section 8.1 suggests that a VSFT may be formulated as a cubic string field theory, with some new choice $\mathcal{Q}$ for the kinetic operator. The choice of $\mathcal{Q}$ will be required to satisfy the following properties:

- The operator $\mathcal{Q}$ must be of ghost number one and must satisfy the conditions (62) that guarantee gauge invariance of the string action.
- The operator $\mathcal{Q}$ must have vanishing cohomology.
- The operator $\mathcal{Q}$ must be universal, namely, it must be possible to write without reference to the brane boundary conformal field theory.

The first condition is unavoidable; the theory must be gauge invariant if it is to be consistent. The second condition is reasonable, but perhaps stronger than needed: all we probably know is that there should be no cohomology at ghost number one, which is the ghost number at which physical states appear. The third constraint is the most stringent one. It implies that VSFT is an intrinsic theory that can be formulated without using an auxiliary D-brane.

The simplest possible choice is $\mathcal{Q} = 0$, which gives the purely cubic version of open string field theory [107]. Indeed, it has long been tempting to identify the tachyon vacuum with a theory where the kinetic operator vanishes because, lacking the kinetic term, the string field gauge symmetries are not spontaneously broken. Nevertheless, there are well-known complications with this identification. The string field shift $\bar{\Phi}$ that relates the cubic to the purely cubic OSFT appears to satisfy $Q\bar{\Phi} = 0$ as well as $\bar{\Phi} * \bar{\Phi} = 0$. The tachyon condensate definitely does not satisfy these two identities. We therefore search for nonzero $\mathcal{Q}$.

We can satisfy the three requirements by letting $\mathcal{Q}$ be constructed purely from ghost operators. In particular we claim that the ghost number one operators

$$\mathcal{C}_n \equiv c_n + (-)^n c_{-n}, \quad n = 0, 1, 2, \cdots \tag{320}$$

satisfy the properties

$$\begin{aligned} &\mathcal{C}_n \mathcal{C}_n = 0, \\ &\mathcal{C}_n (A * B) = (\mathcal{C}_n A) * B + (-1)^A A * (\mathcal{C}_n B), \\ &\langle \mathcal{C}_n A, B \rangle = -(-)^A \langle A, \mathcal{C}_n B \rangle. \end{aligned} \tag{321}$$

The first property is manifest. The last property follows because under BPZ conjugation $c_n \to (-)^{n+1} c_{-n}$. The second property follows from the conservation laws [65]

$$\langle V_3 | (\mathcal{C}_n^{(1)} + \mathcal{C}_n^{(2)} + \mathcal{C}_n^{(3)}) = 0. \tag{322}$$

These conservation laws arise by consideration of integrals of the form $\int dz c(z) \varphi(z)$ where $\varphi(z)(dz)^2$ is a globally defined quadratic differential.

Each of the operators $\mathcal{C}_n$ has vanishing cohomology. To see this note that for each n the operator $B_n = \frac{1}{2}(b_n + (-)^n b_{-n})$ satisfies $\{\mathcal{C}_n, B_n\} = 1$. It then follows that whenever $\mathcal{C}_n \psi = 0$, we have $\psi = \{\mathcal{C}_n, B_n\}\psi = \mathcal{C}_n(B_n \psi)$, showing that ψ is $\mathcal{C}_n$ trivial. Since they are built from ghost oscillators, all $\mathcal{C}_n$'s are manifestly universal.

It is clear from the structure of the consistency conditions that we can take $\mathcal{Q} = \sum_{n=0}^{\infty} a_n \mathcal{C}_n$, where the a_n's are constant coefficients. As we will see below, many properties of the RSZ theory follow simply from the fact that $\mathcal{Q}$ is pure ghost. But, there are some computations that may require a choice of $\mathcal{Q}$ (more on this later). The work of Hata and Kawano [108] gave the clue for the choice of $\mathcal{Q}$ taken by RSZ:

$$\begin{aligned} \mathcal{Q} &= \frac{1}{2i}(c(i) - \bar{c}(i)) = \frac{1}{2i}(c(i) - c(-i)) = \sum_{n=0}^{\infty} (-1)^n \mathcal{C}_{2n}, \\ &= c_0 - (c_2 + c_{-2}) + (c_4 + c_{-4}) - \cdots. \end{aligned} \tag{323}$$

Since the canonical zero-time open string in the complex z-plane is the half-circle $|z| = 1$ that lies on the upper half plane, the operator $c(i)$ represents a ghost insertion precisely at the open string midpoint. This is the most delicate point on the open string given that the three string interaction is a world-sheet with a curvature singularity at the point where the three string midpoints meet. The other operator $c(-i)$ is needed in order that $\mathcal{Q}$ is twist invariant (see the first equation in (75)). With this choice of $\mathcal{Q}$, the string field action is written as

$$S = -K_0 \Big[\frac{1}{2} \langle \Phi, \mathcal{Q}\Phi \rangle + \frac{1}{3} \langle \Phi, \Phi * \Phi \rangle \Big], \tag{324}$$

where the overall normalization K_0 turns out to be infinite. Although the constant K_0 can be absorbed into a rescaling of Ψ, this changes the normalization of $\mathcal{Q}$. We shall instead choose a convenient normalization of $\mathcal{Q}$ and keep the constant K_0 in the action as in eq.(324).

In this VSFT the ansatz was made that any Dp-brane solution takes the factorized form [74]

$$\Phi = \Phi_m \otimes \Phi_g \,, \tag{325}$$

where Φ_g denotes a state obtained by acting with the ghost oscillators on the SL(2,R) invariant vacuum of the ghost CFT, and Φ_m is a state obtained by acting with matter operators on the SL(2,R) invariant vacuum of the matter CFT. Let us denote by $*^g$ and $*^m$ the star product in the ghost and matter sector respectively. Eq.(306) then factorizes as

$$\mathcal{Q}\Phi_g = -\Phi_g *^g \Phi_g \,, \tag{326}$$

and

$$\Phi_m = \Phi_m *^m \Phi_m \,. \tag{327}$$

This last equation is particularly simple: it states that Φ_m is a projector (a projector P in an algebra with product $*$ is an element that satisfies $P * P = P$). The equation for Φ_g appears to be more complicated.

For any string field configuration Φ that satisfies the equation of motion, the action is given by

$$S = -\frac{K_0}{6}\langle \Phi, \mathcal{Q}\Phi\rangle \,, \tag{328}$$

and with the ansatz (325) this becomes

$$S = -\frac{K_0}{6}\langle \Phi_g|\mathcal{Q}\Phi_g\rangle\langle\Phi_m|\Phi_m\rangle \,, \tag{329}$$

Here the inner products are the BPZ ones for the separate matter and ghost conformal field theories. For any static solution, the action is equal to minus the potential energy. If we are describing a Dp-brane, the action is equal to minus the volume of the brane times the tension of the brane.

To proceed further it is assumed that the ghost part Φ_g is universal for all Dp-brane solutions. Under this assumption the ratio of energies associated with two different D-brane solutions with matter parts Φ'_m and Φ_m respectively, is given by:

$$\frac{E'}{E} = \frac{\langle\Phi'_m|\Phi'_m\rangle_m}{\langle\Phi_m|\Phi_m\rangle_m} \,. \tag{330}$$

Thus the ghost part drops out of this calculation. The inner products in the above right-hand side include brane volume factors, which once removed, give us brane tensions. Equation (330) has allowed some important tests of VSFT. If solutions Φ'_m and Φ_m are available, one can calculate the ratio of tensions of D-branes. Since the ratios are known, one has a test of VSFT. The solutions, as mentioned before, are projectors of the star algebra. The D25-brane solution, for example, can be represented by the sliver state $|\Xi\rangle$, which is the first example of a star-algebra projector that was discovered. The sliver state can be constructed for any conformal field theory (a brief discussion is given in the following subsection). Similarly, Dp-brane solutions can be obtained as modified slivers, and numerical verification that the correct ratio of tensions emerges was obtained [74]. Subsequently, and equipped with a better understanding of the star-algebra, Okuyama [109] was able to demonstrate analytically that the correct ratio of tensions emerges.

In a series of stimulating papers [108, 110], Hata, Kawano, and Moriyama, showed that the relationship $2\pi^2 g^2 T_{25} = 1$ between the D25-brane tension and the string coupling can be tested in VSFT without knowledge of the explicit form of the purely ghost $\mathcal{Q}$. In other words, the normalization of the action, the infinite constant K_0, does not feature in the computation. This is easy to see. The D-brane tension, which is proportional to the value of the action evaluated on the sliver solution, is linearly proportional to K_0. In order to calculate the string coupling, Hata and Kawano proposed to look for the tachyon state on the D-brane; this state should appear as a fluctuation around the sliver solution. With this tachyon state, the string coupling g can be obtained as the coupling for three on-shell tachyons. The effective action for the tachyon fluctuation t would take the form

$$K_0\Big(\alpha\,\frac{1}{2}\,t\,(\partial^2+1)t+\frac{1}{3}\,\beta\,t^3\Big)\,, \tag{331}$$

where α and β are calculable finite constants. The field rescaling $t = T/\sqrt{K_0\alpha}$ brings this action to canonical form

$$\frac{1}{2}\,T\,(\partial^2+1)T+\frac{1}{3}\,\frac{\beta}{\sqrt{K_0\alpha}}\,T^3\,, \tag{332}$$

and the string coupling can be read $g = \beta/\sqrt{K_0\alpha}$. Since $T_{25} \sim K_0$, the relation $2\pi^2 g^2 T_{25} = 1$ does not involve K_0. The original computations, however, did not work out, because the tachyon state had been incorrectly identified [111]. In a remarkable work [81], Okawa gave a correct identification of the tachyon state and demonstrated that the relation between the string coupling and the brane tension works out correctly. Still both the string coupling and the brane tension are singular.

It is interesting to wonder what features of VSFT that depend on the particular choice of pure ghost operator $\mathcal{Q}$. It appears that a completely regular definition of the spectrum of strings around D-brane solutions may involve $\mathcal{Q}$. Indeed, Okawa has recently demonstrated that the knowledge of $\mathcal{Q}$ is necessary to produce VSFT solutions that give rise to a string coupling and brane tension both of which are finite [112]. The specific form of $\mathcal{Q}$ may also be needed for the calculation of closed string amplitudes using VSFT. It is clear, however, that the choice in (323) is rather special. We remarked earlier that the equation (326) for the ghost part of the solution is not just a projector equation. It turns out, however, that there is a twist of the ghost CFT of (b, c) in which the antighost becomes a field of dimension one and ghost becomes a field of dimension zero. The new CFT has central charge $c = -2$. If $\mathcal{Q}$ is given by (323), the solution of (326) is simply the sliver state of the twisted conformal field theory. [113]

We conclude this subsection with some comments on regularization and the singular aspects of VSFT. Arguments by Gross and Taylor [68], and by Schnabl (unpublished) indicated that the brane tension associated with VSFT solutions is zero for any finite K_0. Numerical experiments confirm these arguments. A regulation scheme was developed by Gaiotto *et.al*

[113] in which K_0 is replaced by $K_0(a)$, and the gauge-fixed kinetic operator of VSFT is made a-dependent in such a way that for for infinite a the pure ghost operator is recovered. The $K_0(a)$ divergence as $a \to \infty$ is determined from the requirement that the D-brane tension is correctly reproduced. The regulated theory appears to be well defined, but universality is lost in the regulation. On the other hand, the analysis of the regulated equations led to the discovery of another special projector of the star algebra: the butterfly state [113, 64, 114].

We noted in section 8.1 that after a shift to the tachyon vacuum the open string field theory on a D25-brane becomes a cubic string field theory with kinetic operator $\widetilde{Q}$. This operator is not made solely of ghosts. We would expect, however, that the RSZ theory, if fully correct, is field redefinition equivalent to the theory with $\widetilde{Q}$. If we consider the action (309), a homogeneous field redefinition of the type

$$\widetilde{\Phi} = e^K \Phi \,, \tag{333}$$

has special properties if K is a ghost number zero Grassmann even operator that satisfies the following relations

$$\begin{aligned} K(A * B) &= (KA) * B + A * (KB)\,, \\ \langle\, KA, B\,\rangle &= -\langle A, KB\rangle\,. \end{aligned} \tag{334}$$

These properties guarantee that the form of the cubic term is unchanged, and that, after the field redefinition, the action takes the form

$$S(\Phi) = -\frac{1}{g^2}\left[\frac{1}{2}\langle\,\Phi\,,\,\widehat{Q}\,\Phi\rangle + \frac{1}{3}\langle\,\Phi\,,\,\Phi * \Phi\rangle\right], \tag{335}$$

where

$$\widehat{Q} = e^{-K}\,\widetilde{Q}\,e^K\,. \tag{336}$$

It is a good exercise to verify that equations (334) guarantee that $\widetilde{Q}$ satisfies the properties listed in (62). Therefore the new action is consistent.

The operator $\widetilde{Q}$ is, by construction, regular, while $\widehat{Q}$, which we want to be equal to the VSFT operator $\mathcal{Q}$, should be an infinite constant times a ghost insertion at the string midpoint (the infinite constant is necessary because g is finite). A large class of string reparameterizations that leave the open string midpoint invariant can be constructed with operators K that satisfy the relations (334). A reparameterization in which a finite part of the string is squeezed into an infinitesimal neighborhood of the string midpoint will turn a regular $\widetilde{Q}$ that contains a term linear in the ghost field, into an operator $\widehat{Q}$ whose leading term is precisely a divergent ghost insertion at the string midpoint. [113] This happens because the term linear in the ghost field is the term with an operator of lowest possible dimension, and a squeezing transformation, will transform this negative-dimension operator with an infinite factor. It is thus plausible that a singular squeezing transformation relates the string field theory around the tachyon vacuum to the RSZ theory.

8.4 Slivers and projection operators

From the point of view of the RSZ approach to VSFT just discussed, projection operators of the star algebra play a crucial role in the construction of solutions of the theory. Such projection operators may also be useful in understanding solutions in the original Witten theory. Quite a bit of work has been done on constructing and analyzing projectors in the star algebra since the RSZ model was originally proposed. Without going into the technical details, we now briefly review some of the important features of projectors.

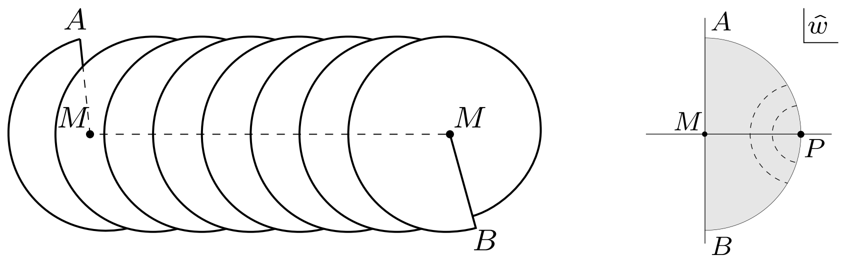


Figure 9: The sliver appears as a cone with infinite excess angle– namely, an infinite helix. The segments AM and BM represent the left-half and the right-half of the string. The local coordinate patch, represented by the shaded half disk shown to the right, must be glued in to form the complete surface.

The first matter projector which was explicitly constructed is the "sliver" state. This state was identified as a conformal field theory surface state by Rastelli and Zwiebach [65]. As such, there is a surface associated with the state: a disk with one puncture on the boundary and a specified local coordinate at the puncture. This conformal field theory picture gives a complete state; it includes both the matter and the ghost part of the state. Moreover, the state can be constructed for any conformal field theory:

$$|\Xi\rangle = \exp\Big(-\frac{1}{3}L_{-2} + \frac{1}{30}L_{-4} - \frac{11}{1890}L_{-6} + \frac{34}{467775}L_{-8} + \cdots\Big)|0\rangle\,. \tag{337}$$

The geometrical picture of the sliver state is shown in figure 9. The full punctured disk is the glued surface obtained by attaching the infinite helix and the coordinate patch, which carries the puncture P. There are many alternative pictures of the sliver.

To understand why the sliver state squares to itself one must have a picture of star multiplication for surface states. A full discussion [115] would take too long, but the rough idea is easily explained. The sliver state is essentially the limit $\lim_{n\to\infty}(|0\rangle)^n$, where multiplication is performed via the star product. A surface state in a BCFT can be viewed (by excising the coordinate patch) as a disk whose boundary has two parts: a part in which the boundary condition that defines the BCFT is imposed, and a part which represents an open string. To star-multiply two surface states, one glues the right-half of the string in the first surface to the left-half of the string in the second surface; the resulting surface is the surface that represents the star product. A particularly simple class of surface states are sector states or wedge states. One such state $\mathcal{R}_\alpha$ is shown to the left of figure 10. The BCFT boundary

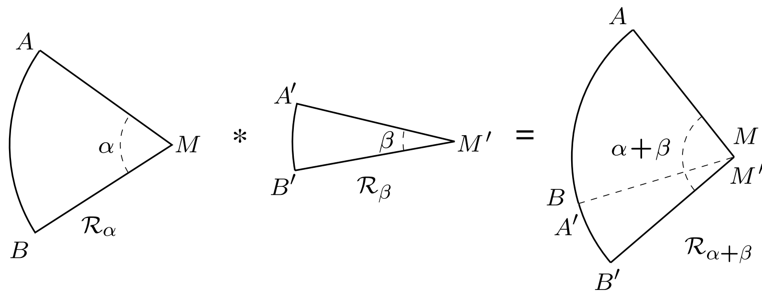


Figure 10: The star multiplication of a sector state with angle α to a sector state with angle β gives a sector state with angle $\alpha+\beta$. Sector states are just another presentation of wedge states.

condition applies to the curved boundary of the sector. The radial segment AM is the left-half of the open string and the radial segment MB is the right-half of the open string. The sector state is defined by the angle α at the string midpoint M. In the figure we show the multiplication of $\mathcal{R}_\alpha$ and $\mathcal{R}_\beta$. The result is a sector state $\mathcal{R}_{\alpha+\beta}$ with total angle $\alpha+\beta$. The sliver state Ξ is the wedge state $\mathcal{R}_\infty$ with infinite angle. It is then clear that the star product of two slivers is still a wedge state of infinite angle, and thus also a sliver. The state obtained in the limit when the angle is equal to zero is in fact the identity state of the star algebra. It is manifestly clear that the product of any surface state with the identity gives the surface state. The identity state can also be written as an exponential of Virasoro operators acting on the vacuum. In fact, as mentioned in section 8.2, a very curious result was found [63]:

$$\begin{aligned} |\mathcal{I}\rangle &= \left(\prod_{n=2}^{\infty} \exp\left\{-\frac{2}{2^n} L_{-2^n}\right\}\right) e^{L_{-2}}|0\rangle \\ &= \ldots \exp(-\frac{2}{2^3} L_{-2^3}) \exp(-\frac{2}{2^2} L_{-2^2}) \exp(L_{-2})|0\rangle \,, \end{aligned} \tag{338}$$

with the Virasoro operators of higher level stacking to the left. We thus confirm that the identity is also a Virasoro descendent of the vacuum.

In an independent construction, Kostelecky and Potting [116] constructed a state Ψ_m of the matter sector of the D25-brane BCFT that squared to itself (up to a proportionality constant). The construction used the oscillator language. This matter state takes the form of a squeezed state

$$|\Psi_m\rangle = \mathcal{N} \exp\left[\frac{1}{2}\, a^\dagger \cdot S \cdot a^\dagger\right] |0\rangle \,. \tag{339}$$

By requiring that such a state satisfy the projection equation $\Psi \star \Psi = \Psi$, and by making some further assumptions about the nature of the state, an explicit formula for the matrix S was found in terms of the matrix X from (164) [116]. Evidence quickly emerged that the state constructed by these authors is the matter sector of the sliver state, and a proof was given by Okuda [117].

There are many other projectors that also have a simple picture as surface states [64, 114, 118]. In these projectors, the open string midpoint approaches (or even coincides with) the boundary of the surface where the boundary condition is applied. One particularly useful projector, which arises in the numerical solution of VSFT, is the so-called *butterfly* state $\mathcal{B}$. This is a very interesting state, whose picture is shown in Figure 11. When one glues two butterfly surfaces in the manner required by star-multiplication, the resulting surface does not appear to be, at first sight, another butterfly. Nevertheless, the resulting surface is conformally equivalent to a butterfly, and this is, in fact, all that is needed in order to have a projector. It has been demonstrated that the butterfly is the state that can be represented as the tensor product $|0\rangle \otimes |0\rangle$, where $|0\rangle$ is the vacuum of the half-string state space [114]. Generally, any state of the form $|a\rangle \otimes |a\rangle$ where $|a\rangle$ is the same state in the left and right half-string Fock spaces is a projector [67]. The butterfly has a remarkably simple expression as a Virasoro descendent of the vacuum

$$|\mathcal{B}\rangle = \exp\Big(-\frac{1}{2}L_{-2}\Big)|0\rangle\,. \tag{340}$$

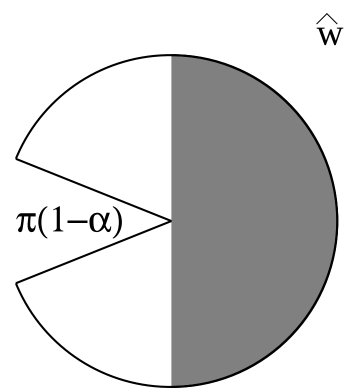


Figure 11: The butterfly state arises in the limit where $\alpha \to 1$ and the angle indicated in the figure vanishes.

Projectors have many properties which are reminiscent of D-branes. This relationship between projection operators and D-branes is familiar from noncommutative field theory, where projectors also play the role of D-brane solitons [119]. This connection becomes quite concrete in the presence of a background B field [120, 121]. In the RSZ theory, states that describe an arbitrary but fixed configuration of D-branes are constructed by tensoring the matter projector for the appropriate BCFT with a fixed ghost state that satisfies the ghost equation of motion (326). Particular projectors like the sliver can be constructed which are localized in any number of space-time dimensions, corresponding to the codimension of a D-brane. Under gauge transformations, a rank one projector can be rotated into an orthogonal rank one projector, so that configurations containing multiple branes can be constructed as higher rank projectors formed from the sum of orthogonal rank one projectors [122, 67]. This gives a very suggestive picture of how arbitrary D-brane configurations can be constructed in string field theory.

While this picture is quite compelling, however, there are some technical obstacles which make this still a somewhat incomplete story. In the RSZ model, singularities appear due to the separation of the matter and ghost sectors. Moreover, projectors are, in general, somewhat singular states. For example, the matrix S associated to the matter part of the sliver state has eigenvalues of ± 1 for any Dp-brane [120, 114]. Such eigenvalues cause these states to be non-normalizable elements of the matter Fock space In the Dirichlet directions, this lack of normalizability occurs because the state is essentially localized to a point and is analogous to a delta function. In the Neumann directions, the singularity manifests as a "breaking" of the strings composing the D-brane, so that the functional describing the projector state is a product of a function of the string configurations on the left and right halves of the string, with no connection mediated through the midpoint. These geometric singularities seem to be generic features of the matter part of any projector, not just the sliver state [64, 114]. The singular geometric features of projectors, which can be traced to the fact that the open string midpoint approaches the boundary, makes certain calculations in the RSZ theory somewhat complicated, as all singularities must be regulated. Singularities do not seem to appear in the Witten theory, where the BRST operator and the numerically calculated solutions seem to behave smoothly at the string midpoint. On the other hand, it may be that further study of the projectors will lead to analytic progress on the Witten theory, as discussed in a recent paper by Okawa [123].

8.5 Closed strings in open string theory

We have discussed in earlier sections the fact that open string field theory, formulated on the background of a certain BCFT appears to capture many other open string backgrounds as solutions of the theory. Apart from its singular features, the RSZ theory admits any BCFT as a solution of the theory. One important question remains: Can closed string backgrounds be incorporated in open string field theory? The question can be answered both in the context of OSFT and in the context of the RSZ model. As we will discuss, there is very little concrete evidence as yet that this can be done in any of the two approaches. We therefore ask a simpler question: Can closed string states be seen in open string field theory? The answer here is yes, both in OSFT, and in VSFT (modulo the usual singularities), although so far this has been understood only in certain limited contexts.

As has been known since the earliest days of the subject, closed strings appear as poles in perturbative open string scattering amplitudes. This was demonstrated explicitly for Witten's theory by exhibiting the closed string poles arise in the one-loop 2-point function [124] (although in this calculation, spurious poles also appear which complicate the interpretation). More recently, in a similar calculation the closed string tadpole generated by the D-brane was identified in the one-loop open string 1-point function [125]. While in principle this type of argument can be used to construct all on-shell closed string amplitudes through factorization, it is much less clear how to think of asymptotic or off-shell closed string states in this context. If Witten's theory is well-defined as a quantum theory, it would follow from unitarity that the closed string states should also arise in some natural sense as asymptotic states of the quantum open string field theory. It is currently rather unclear, however, whether, and if so how, this might be realized. There are subtleties in the quantum formulation of the theory which have never completely been resolved [59, 125], although most of the problems of the quantum theory seem to be generated by the closed string tachyon, and may be absent in a supersymmetric theory. Both older SFT literature [126, 127] and recent work [128, 120, 113, 129] have suggested ways in which closed strings might be incorporated into the open string field theory, but a definitive resolution of this question is still not available.

In the RSZ model, one description of *on-shell* closed string states is reasonably natural [113, 129, 130, 131] and scattering amplitudes have been computed [132, 133]. For each on-shell closed string vertex operator V one can construct a gauge-invariant open string state $\mathcal{O}_V(\Phi)$, where Φ is the open string field, and the gauge invariance is the open string gauge invariance. The world-sheet picture of the state is that of an amputated semi-infinite strip whose edge represents the open strings, the two halves of which are glued and the closed string operator is inserted at the conical singularity. Given a set of gauge invariant operators associated with a set of on-shell closed string vertex operators, the RSZ correlator of the gauge invariant operators appears to give, up to proportionality factors that need regulation, the on-shell closed string amplitude on a surface *without* boundaries. This result uses a nontrivial and unusual decomposition of the moduli space of Riemann surfaces without boundaries [113]. The decomposition, is related to, but distinct from the one used in Witten's theory to cover the moduli space of Riemann surfaces that have at least one boundary. Other decompositions have been discussed by Drukker [131].

If it were possible to encode *off-shell* physics naturally into open string field theory it would be reasonable to hope that closed string backgrounds could be changed by suitable expectation values of open string fields although this would presumably be a subtle effect in the quantum theory, and difficult to compute explicitly. Attaining a description of the full closed string landscape [134] using quantum OSFT is clearly an optimistic scenario, but it need not be farfetched; it may represent an extension of the AdS/CFT correspondence, in which the CFT side is changed from SYM into the full open string field theory. If, as it may be, it turns out to be that the closed string sector of the theory is encoded in a singular fashion in OSFT, one may be better off directly working with closed string field theory [27], or with open/closed string field theory [31]. Because of the nonpolynomiality of these theories, it is not known at present if level expansion can be used to extract nonperturbative information. At any rate, it would be useful to have a clear picture of how far one can incorporate closed string physics from the open string point of view. Even if this cannot be realistically achieved in our current models of SFT, understanding the difficulties involved may help us in our search for a better formulation of the theory.

9 Conclusions

The work described in these lectures has brought the understanding of string field theory to a new level. We now have fairly conclusive evidence that open string field theory can successfully describe distinct vacua with very different geometrical properties, which are not related to one another through a marginal deformation. The resulting picture, in which a complicated set of degrees of freedom defined primarily through an algebraic structure, can produce different geometrical backgrounds as different solutions of the equations of motion, represents an important step beyond perturbative string theory. Such a framework is clearly necessary to discuss questions of a cosmological nature in string theory. For such questions, however, one must generalize from the work described here in which the theory describes distinct *open* string backgrounds, to a formalism where different *closed* string backgrounds also appear as solutions of the equations. Ideally, we would like to have a formulation of string/M-theory in which all the currently understood vacua can arise in terms of a single well-defined set of degrees of freedom.

It is not yet clear, however, how far it is possible go towards this goal using the current formulations of string field theory. It may be that the correct lesson to take from the work described here is simply that there *are* nonperturbative formulations in which distinct vacua can be brought together as solutions of a single classical theory, and that one should search for some deeper fundamental algebraic formulation where geometry, and even the dimension of space-time emerge from the fundamental degrees of freedom in the same way that D-brane geometry emerges from the degrees of freedom of Witten's open string field theory. A more conservative scenario, however, might be that we could perhaps use the current framework of string field theory, or some limited refinement thereof, to achieve this goal of providing a universal nonperturbative definition of string theory and M-theory. Following this latter scenario, we propose here a series of questions aimed at continuing the recent developments in open string field theory as far as possible towards this ultimate goal. It is not certain that this research program can be carried to its conclusion, but it will be very interesting to see how far open string field theory can go in reproducing important nonperturbative aspects of string theory.

There are, in our mind, two very important concrete problems related to Witten's string field theory that so far have resisted solution:

1) Finding an analytic description of the tachyonic vacuum. Despite several years of work on this problem, great success with numerical approximations, and some insight from the RSZ vacuum string field theory model, we still have no closed form expression for the string field Φ_0 which represents the tachyon vacuum in Witten's open string field theory. It seems almost unbelievable that there is not some elegant analytic solution to this problem. An analytic solution would almost certainly greatly enhance our understanding of this theory and would lead to other significant advances.

2) Finding certain open string backgrounds as solutions of open string field theory. As discussed in section 7.6, we do not know how to obtain a background with multiple

D-branes starting with a background with one D-brane. Nor we know how to obtain the background which represents a D0-brane using the background of a D1-brane with lower energy. It is currently unclear whether the obstacles to finding these vacua are conceptual or technical.

There are other questions that are probably important to the future development of string field theory. These represent, in our opinion, subjects that merit investigation:

1) Is there a regular formulation of VSFT ? Such a version of the theory may have further similarities with BSFT and could turn out to be a complete and flexible formulation of open string field theory.

2) How do closed string backgrounds appear in open string field theory? While OSFT and VSFT appear to give somewhat singular/intractable descriptions of closed string physics, some better understood, or new, version of open string theory might provide a tractable description of closed string physics. Another possibility is that closed string fields are needed in addition to open string fields; this is the case in light-cone open string field theory and in covariant open/closed string field theory.

3) What are the new features of superstring field theory? The status of the tachyon conjectures for the superstring has been reviewed by Ohmori [135]. The large set of symmetries of superstring theory makes them, in many cases, more tractable than bosonic string theories. Nevertheless, as of yet, there is no clear sense in which superstring field theory is simpler than bosonic string field theory [136]. There are also significant conceptual problems that have not allowed a formulation of vacuum superstring field theory [137].

4) How do we describe time-dependent tachyon dynamics? String field theory gives clear and concrete evidence for the Sen conjectures. Although we have not studied this subject in the present review, there is much interest in the process by which the tachyon rolls from the unstable critical point down to the tachyon vacuum. [6] In fact, the early attempts to describe the rolling of the tachyon in Witten's string field theory [138, 139] appear to be in contradiction with the results that follow from conformal field theory.

It is challenging to imagine a single set of degrees of freedom which could encode, in different phases, all the possible string backgrounds we are familiar with, including those associated with M-theory. In principle, a nonperturbative background-independent formulation of type II string theory should allow one to take the string coupling to infinity in such a way that the fundamental degrees of freedom of the theory remain at some finite point in the configuration space. This would lead to the vacuum associated with M-theory in flat space-time. It would be quite remarkable if this can be achieved in the framework of string field theory. Given the nontrivial relationship between string fields and low-energy effective degrees of freedom, such a result need not be farfetched. If this picture could be successfully implemented, it would give a very satisfying representation of the complicated network of dualities of string and M-theory in terms of a single underlying set of degrees of freedom.