Contents

1 Introduction

Historically, the Dirac operator was put forward as a result of making the Schrödinger equation in quantum mechanics compatible with the special theory of relativity. In modern physical theories, matter at very small distances, or equivalenty, very high-energies, is made of elementary particles whose interactions are the four fundamental forces in nature: gravity, electromagnetic theory, weak-nuclear force and strong-nuclear force.

Locally the four-dimensional space-time is Minkowskian and physical states are invariant under rotations (Lorentz transformations) and translations. The physical states fall into representations of the Poincare algebra. Particles will be characterized by their mass and spin. Matter particles have spins 0 and $\frac{1}{2}$ obeying respectively Klein-Gordon and Dirac equations. Particles with spin-1 are the mediators of gauge interactions, e.g. photons mediate the electromagnetic interactions between electrons while all particles exchange gravitons for the gravitational interactions.

The general plan is to first study representations of the Poincare algebra and place particles in these representations, and to form physical states obeying these symmetries and write Lorentz-invariant interactions. The Poincare algebra is then generalized to the supersymmetry algebra. We find representations of the supersymmetry algebra and construct interactions invariant under this new symmetry. For more details the reader may consult the following references [1], [2] , [3] , [4], [5] , [6] , [7] , [8] .

2 Poincare group

Einstein's special theory of relativity states the equivalence of certain inertial frames of reference. The coordinates x^μ in an inertial frame satisfy

$$ds^2 = dx^\mu \eta_{\mu\nu} dx^\nu = dx'^\mu \eta_{\mu\nu} dx'^\nu,$$

where x^μ and x'^μ are the coordinates of the physical states in different frames and $\eta_{\mu\nu} = diag\,(1,-1,-1,-1)$ is the Minkowski metric. The transformations $x^\mu \to x'^\mu = a^\mu + \Lambda^\mu{}_\nu x^\nu$ keeps the distance invariant provided that

$$\Lambda^\mu{}_\kappa \eta_{\mu\nu} \Lambda^\nu{}_\lambda = \eta_{\kappa\lambda}. \tag{1}$$

Taking the determinant on both sides gives $(\det\Lambda)^2 = 1$ or $\det\Lambda = \pm 1$. In particular $\Lambda^0{}_0 \eta_{00} \Lambda^0{}_0 + \Lambda^i{}_0 \eta_{ij} \Lambda^j{}_0 = \eta_{00}$, or $\left(\Lambda^0{}_0\right)^2 = 1 + \left(\Lambda^i{}_0\right)^2$. This implies that $\Lambda^0{}_0 \geq 1$, or $\Lambda^0{}_0 \leq -1$ and transformations are divided into four categories. The entire set of Λ comprises the homogeneous Lorentz group. The orthocronous Lorentz transformations are charecterised by the condition $\Lambda^0{}_0 \geq 1$ so that every positive time-like vector transforms into positive time-like vector. If $\det\Lambda = 1$ then this gives a group of restricted homogeneous Lorentz-transformations. These are the only ones that could be obtained by a continuous transformation of the identity. This is because it is not possible to continously transform states with $\det\Lambda > 0$ to states with $\det\Lambda < 0$, or states with $\Lambda^0{}_0 \geq 1$ to states with $\Lambda^0{}_0 \leq -1$. Therefore these states correspond to an $SO(3,1)$ group. The translations a^μ form a seperate group, an abelian one, defined as $T\,(4)$. We write

$$\begin{aligned} x' &= \Lambda x + a \equiv g\,(a,\Lambda)\,x, \\ x'' &= \Lambda' x + a' = \Lambda'\,(\Lambda x + a) + a' = \Lambda'\Lambda x + (\Lambda' a + a')\,. \end{aligned}$$

But $x'' = g\,(a',\Lambda')\,x' = g\,(a',\Lambda')\,g\,(a,\Lambda)\,x \equiv g\,(a'',\Lambda'')\,x$, which implies

$$g\,(a',\Lambda')\,g\,(a,\Lambda) = g\,(\Lambda' a + a', \Lambda'\Lambda)\,.$$

The subgroup $SO(3,1)$ is defined by $g(0,\Lambda')\,g(0,\Lambda)=g(0,\Lambda'\Lambda)$ and the abelian subgroup T^4 by $g(a',1)\,g(a,1)=g(a+a',1)$. The inverse transformation is determined by the two conditions $\Lambda' a+a'=0$, and $\Lambda'\Lambda=1$ which implies

$$g^{-1}(a,\Lambda)=g\left(-\Lambda^{-1}a,\Lambda^{-1}\right).$$

3 Poincare Lie algebra

We write $\Lambda^{\mu}_{\ \nu}=\delta^{\mu}_{\nu}+\zeta^{\mu}_{\ \nu}$ where $\zeta^{\mu}_{\ \nu}$ is infinitesimal. This equation implies that $\zeta_{\mu\nu}=\eta_{\mu\lambda}\zeta^{\lambda}_{\ \nu}=-\zeta_{\nu\mu}$ is antisymmetric. Assume a^{μ} is infinitesimal then we need four essential parameters for translation and six essential parameters for rotations

$$U(a,\Lambda)=1+i\delta a^{\mu}P_{\mu}-\frac{i}{2}\zeta^{\mu\nu}J_{\mu\nu}.$$

The commutation relations of the Lie algebra can be obtained via the usual basis of scalar functions

$$U(a,\Lambda)f(x)=f\left(U^{-1}(a,\Lambda)\,x\right)=f\left(\Lambda^{-1}(x-a)\right).$$

By going to the infinitesimal limit we get

$$\begin{aligned}\left(1+i\delta a^{\mu}P_{\mu}-\frac{i}{2}\zeta^{\mu\nu}J_{\mu\nu}\right)f\left(x^{\lambda}\right)&=f\left(x^{\lambda}-\delta a^{\lambda}-\zeta^{\lambda}_{\ \kappa}x^{\kappa}\right)\\&=f\left(x^{\lambda}\right)-\left(\delta a^{\lambda}+\zeta^{\lambda}_{\ \kappa}x^{\kappa}\right)\partial_{\lambda}f(x)+\cdots\end{aligned}$$

This implies that

$$\begin{aligned}P_{\mu}&=i\frac{\partial}{\partial x^{\mu}}\equiv i\partial_{\mu},\\J_{\mu\nu}&=\left(x_{\mu}P_{\nu}-x_{\nu}P_{\mu}\right).\end{aligned}$$

These generators obey the Lie algegra

$$\begin{aligned}\left[P_{\mu},P_{\nu}\right]&=0,\\\left[J_{\mu\nu},P_{\lambda}\right]&=i\left(\eta_{\nu\lambda}P_{\mu}-\eta_{\mu\lambda}P_{\nu}\right),\\\left[J_{\mu\nu},J_{\rho\sigma}\right]&=-i\left(\eta_{\mu\rho}J_{\nu\sigma}-\eta_{\nu\rho}J_{\mu\sigma}-\eta_{\mu\sigma}J_{\nu\rho}+\eta_{\nu\sigma}J_{\mu\rho}\right).\end{aligned}$$

By exponntiating, a general finite transformation is expressible in the form $U(a,\Lambda)=\exp\left(i\delta a^{\mu}P_{\mu}-\frac{i}{2}\zeta^{\mu\nu}J_{\mu\nu}\right)$. The Casimir invariants (operators that commute with the generators of the Lie algebra) are $P^2=P^{\mu}\eta_{\mu\nu}P^{\nu}$ and $W^2=W^{\mu}\eta_{\mu\nu}W^{\nu}$ where $W_{\mu}=-\frac{1}{2}\epsilon_{\mu\nu\rho\sigma}P^{\nu}J^{\rho\sigma}$. One can easily verify that

$$\begin{aligned}\left[P^2,P_{\mu}\right]&=0,\quad\left[P^2,J_{\mu\nu}\right]=0,\\\left[W^2,P_{\mu}\right]&=0,\quad\left[W^2,J_{\mu\nu}\right]=0.\end{aligned}$$

If $P^2 \neq 0$ we can use the spin pseudo-vector $S_\mu = \frac{1}{\sqrt{P^2}} W_\mu$. The Casimir eigenvalues of P^2 and W^2 (or $j\,(j+1)$ associated with S^2) are used to specify irreducible representations:

$$\begin{aligned} P_\mu \left| p, j, \lambda \right\rangle &= p_\mu \left| p, j, \lambda \right\rangle, \\ W^2 \left| p, j, \lambda \right\rangle &= W^2\,(j) \left| p, j, \lambda \right\rangle, \\ W_0 \left| p, j, \lambda \right\rangle &= W_0\,(j) \left| p, j, \lambda \right\rangle. \end{aligned}$$

There are four distinct cases. i) $P^2 > 0$, ii) $P^2 = 0$, iii) $P^2 < 0$, iv) $P_\mu = 0$. If $P^2 \neq 0$ then we can take $P_1 = P_2 = 0$. The little group leaving this choice invariant contains $J_3 = J_{12}$, so we can associate λ with integer or $\frac{1}{2}$-integer eigenvalues of J_3 :

$$J_3 \left| \widehat{p}, j, \lambda \right\rangle = \lambda \left| \widehat{p}, j, \lambda \right\rangle,$$

so that λ is always discrete.

If $P^2 = m^2 > 0$ we choose the rest frame

$$\begin{aligned} P_\mu &= \pm\,(m, 0, 0, 0)\,, \\ W_\mu &= \pm m \left(0, \overrightarrow{J}\right), \quad J_{ij} = \epsilon_{ijk} J^k, \\ S_\mu &= \pm \left(0, \overrightarrow{J}\right). \end{aligned}$$

The little group is $O(3)$ generated by $[J_i, J_j] = \epsilon_{ijk} J^k$ and is described by the state vectors

$$\left| j, \lambda \right\rangle, \quad \lambda = -j, \cdots, j, \quad j = 0, \frac{1}{2}, 1, \cdots$$

where $W^2 = -m^2 j(j+1)$.

If $P^2 = 0$, we can take $P_\mu = (w, 0, 0, w)$, and $W_\mu = w(J_3, L_1, L_2, J_3)$ where $L_1 = J_1 + K_2$, $L_2 = J_2 - K_1$ and $J_{0i} = K_{0i}, \quad i = 1, 2, 3$. The generators L_1 L_2 and J_3 satisfy the commutation relations

$$[L_1, L_2] = 0, \quad [J_3, L_1] = -iL_2, \quad [J_3, L_2] = -iL_1.$$

The relations $P^2 = 0$, $W^2 = 0$ and $P^\mu W_\mu = 0$ imply that P^μ and W^μ are parallel vectors and therefore we can write

$$W_\mu = \lambda P^\mu,$$

where λ is the helicity. This implies that $W^2 = -w^2 L^2$ and unitary representations are labelled by $\left| l, \lambda \right\rangle$.

For the Lorentz group we can write

$$N_i = \frac{1}{2}(J_i + iK_i), \quad N_i^\dagger = \frac{1}{2}(J_i - iK_i),$$

where these generators satisfy the commutation relations

$$\begin{aligned} \left[N_i, N_j^\dagger\right] &= 0, \\ [N_i, N_j] &= i\epsilon_{ijk} N^k, \\ \left[N_i^\dagger, N_j^\dagger\right] &= i\epsilon_{ijk} N^{\dagger k}. \end{aligned}$$

The Casimir operators are given, in terms of the generators N_i and $N_i^\dagger$, by $N_i N_i$ and $N_i^\dagger N_i^\dagger$ with eigenvalues $n(n+1)$ and $m(m+1)$ respectively. The N_i and $N_i^\dagger$ obey independently Lie algebras of $SU(2)$. We can label representations with (m,n) where $m, n = 0, \frac{1}{2}, 1, \frac{3}{2}, \cdots$ are the separate $SU(2)$ numbers. The two $SU(2)$'s are not independent. The spin representation are given by $m+n$. As examples we have:

- $(0,0)$ is spin zero and is the scalar representation.
- $(\frac{1}{2},0)$ or $(0,\frac{1}{2})$ is the spin-$\frac{1}{2}$ representation for Weyl spinors, while Dirac spinors are represented by $(\frac{1}{2},0)\oplus(0,\frac{1}{2})$.
- $(\frac{1}{2},\frac{1}{2}) = (\frac{1}{2},0)\otimes(0,\frac{1}{2})$ is the spin-1 representation.

We represent $(\frac{1}{2},0)$ and $(0,\frac{1}{2})$ by two-component complex spinors ψ_α and $\overline{\psi}^{\dot{\alpha}}$ which transform under Lorentz transformation as

$$\psi'_\alpha = M_\alpha^{\ \beta}\psi_\beta, \quad \overline{\psi}^{\dot{\alpha}} = \left(M^{*-1}\right)^{\dot{\alpha}}_{\ \dot{\beta}}\overline{\psi}^{\dot{\beta}}$$

This can be denoted by $\psi \to M\psi$ and $\overline{\psi} \to \left(M^\dagger\right)^{-1}\overline{\psi}$. We identify generators J_i with $\frac{1}{2}\sigma_i$ and K_i with $-\frac{i}{2}\sigma_i$. We can therefore write

$$M = e^{\frac{i}{2}\sigma_i\left(w^i - iv^i\right)}, \quad \left(M^\dagger\right)^{-1} = e^{\frac{i}{2}\sigma_i\left(w^i + iv^i\right)}.$$

We also introduce the 2×2 matrices $(\sigma^\mu)_{\alpha\dot{\beta}}$ defined by $\left(\sigma^0\right)_{\alpha\dot{\beta}} = I_2$, the identity matrix and $\left(\sigma^i\right)_{\alpha\dot{\beta}} = -\sigma_i$ are the Pauli matrices. The matrix P is defined by $P = P_\mu(\sigma^\mu)$ and transforms under Lorentz transformations according to $P' = MPM^\dagger$ or equivalently $P_\mu\sigma^\mu \to P_\mu M\sigma^\mu M^\dagger$. We also define

$$\overline{\sigma}^{\mu\dot{\alpha}\beta} = \epsilon^{\dot{\alpha}\dot{\gamma}}\epsilon^{\beta\delta}\sigma^\mu_{\ \delta\dot{\gamma}}.$$

These satisfy the proporties

$$\left(\sigma^\mu\overline{\sigma}^\nu + \sigma^\nu\overline{\sigma}^\mu\right)_\alpha^{\ \beta} = 2\eta^{\mu\nu}\delta_\alpha^\beta,$$
$$\left(\overline{\sigma}^\mu\sigma^\nu + \overline{\sigma}^\nu\sigma^\mu\right)_{\dot{\alpha}}^{\ \dot{\beta}} = -2\eta^{\mu\nu}\delta_{\dot{\alpha}}^{\dot{\beta}}.$$

From these matrices we can construct representations of the Lorentz algebra. They are given by

$$\sigma^{\mu\nu\ \beta}_{\ \ \alpha} = \left(\sigma^\mu\overline{\sigma}^\nu - \sigma^\nu\overline{\sigma}^\mu\right)_\alpha^{\ \beta},$$
$$\overline{\sigma}^{\mu\nu\ \dot{\beta}}_{\ \ \ \dot{\alpha}} = \left(\overline{\sigma}^\mu\sigma^\nu - \overline{\sigma}^\nu\sigma^\mu\right)_{\dot{\alpha}}^{\ \dot{\beta}}.$$

A Dirac spinor takes the form

$$\psi = \begin{pmatrix} \chi_\alpha \\ \overline{\lambda}^{\dot{\alpha}} \end{pmatrix},$$

and the Dirac gamma matrices are defined by

$$\gamma^\mu = \begin{pmatrix} 0 & \sigma^\mu \\ \overline{\sigma}^\mu & 0 \end{pmatrix},$$

which satisfy the anticommutation relations $\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu} I_4$. This defines a Clifford algebra. The representation of the Lorentz algebra in terms of the gamma matrices is given by

$$S^{\mu\nu} = \frac{i}{4}\left[\gamma^\mu, \gamma^\nu\right].$$

Let $\Lambda_{\frac{1}{2}} = \exp\left(-\frac{i}{2}\zeta_{\mu\nu}S^{\mu\nu}\right)$ then $\Lambda_{\frac{1}{2}}^{-1}\gamma^\mu\Lambda_{\frac{1}{2}} = \Lambda^\mu{}_\nu\gamma^\nu$ where $\Lambda^\mu{}_\nu = \delta^\nu_\mu + \zeta^\mu{}_\nu$. These proporties will be used in the next section when we derive the Dirac equation.

4 Dynamical equations for free and interacting fields

4.1 Spin-0 : Klein-Gordon field

The Hamiltonian in relativistic dynamics has eigenvalues E which can be determined from the relation $P_\mu P^\mu = m^2$. This relation implies that $P_0^2 = \overrightarrow{P}.\overrightarrow{P} + m^2 = E^2$, where we have set the velocity of light c to 1. Using the quantum mechanical correspondence $P_\mu = \frac{\hbar}{i}\frac{\partial}{\partial x^\mu}$ and setting $\hbar = 1$) and acting with the operator $\left(P_\mu P^\mu - m^2\right)$ on the scalar field ϕ gives

$$\left(\eta^{\mu\nu}\partial_\mu\partial_\nu + m^2\right)\phi = 0,$$

which is the Klein-Gordon equation. This equation could be derived from a varriational principle using the action

$$I_0 = \int d^4x L = \frac{1}{2}\int d^4x \left(\eta^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - m^2\phi^2\right).$$

The Euler-Lagrange equation obtained by demanding that $\delta I = 0$, with the field ϕ vanishing at the boundary, is given by

$$\frac{\delta L}{\delta\phi} = \partial_\mu\left(\frac{\delta L}{\delta\partial_\mu\phi}\right),$$

and this implies the Klein-Gordon equation.

4.2 Spin-$\frac{1}{2}$: Dirac field

The Schrödinger equation is non-relativistic and is linear in time-derivatives but quadratic in space-derivatives

$$i\frac{\partial\psi}{\partial t} = -\frac{\hbar^2}{2m}\overrightarrow{\nabla}^2\psi.$$

The relativistic equation should be linear in time and one must find a first order differential equation of the form

$$i\hbar\frac{\partial\psi}{\partial t} = \frac{\hbar c}{i}\alpha^i\frac{\partial\psi}{\partial x^i} + \beta m\psi,$$

where α^i and β are 4×4 matrices. The square of this equation should yield an equation of the Klein-Gordon type. This is possible if α^i and β satisfy the proporties

$$\alpha^i\alpha^j + \alpha^j\alpha^i = 2\delta^{ij}, \quad \beta^2 = 1, \quad \alpha^i\beta + \beta\alpha^i = 0.$$

Multiplying both sides of the equation by β and defining $\gamma^0 = \beta$, $\gamma^i = \beta\alpha^i$ gives

$$\left(i\gamma^\mu\partial_\mu - m\right)\psi = 0.$$

The matrices γ^μ satisfy the anticommutation relations $\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu}$ and can be identified with the gamma matrices defined in the previous section. This is the Dirac equation. One can easily verify that it is Lorentz covariant

$$\begin{aligned}\left(i\gamma^\mu\partial_\mu - m\right)\psi &\to \left(i\gamma^\mu\Lambda_\mu^{-1\nu}\partial_\nu - m\right)\Lambda_{\frac{1}{2}}\psi\left(\Lambda^{-1}x\right)\\ &= \Lambda_{\frac{1}{2}}\Lambda_{\frac{1}{2}}^{-1}\left(i\gamma^\mu\Lambda_\mu^{-1\nu}\partial_\nu - m\right)\Lambda_{\frac{1}{2}}\psi\left(\Lambda^{-1}x\right)\\ &= \Lambda_{\frac{1}{2}}\left(i\gamma^\mu\partial_\mu - m\right)\psi\left(\Lambda^{-1}x\right),\end{aligned}$$

where we have used the property $\Lambda_{\frac{1}{2}}^{-1}\gamma^\mu\Lambda_{\frac{1}{2}} = \Lambda^\mu{}_\nu\gamma^\nu$ derived in the last section.

The action

$$I_{\frac{1}{2}} = \int d^4x\overline{\psi}(x)\left(i\gamma^\mu\partial_\mu - m\right)\psi(x),$$

is Lorentz invariant because $\overline{\psi} \to \overline{\psi}\Lambda_{\frac{1}{2}}^{-1}$. In the special case when $m = 0$, the Dirac equation simplifies to

$$\begin{pmatrix} 0 & i\sigma^\mu\partial_\mu \\ i\overline{\sigma}^\mu\partial_\mu & 0 \end{pmatrix}\begin{pmatrix} \psi_L \\ \psi_R \end{pmatrix} = 0,$$

and this splits into two independent equations, one for the left-handed component ψ_L and the other for the right handed component ψ_R

$$i\sigma^\mu\partial_\mu\psi_L = 0, \quad i\overline{\sigma}^\mu\partial_\mu\psi_R = 0.$$

4.3 Spin-1: Maxwell field

We are interested in massless vectors, in particular, the photon which obeys Maxwell's equations

$$\begin{aligned}\overrightarrow{\nabla}\cdot\overrightarrow{E} = \rho, \quad \overrightarrow{\nabla}\times\overrightarrow{E} = -\frac{\partial\overrightarrow{B}}{\partial t},\\ \overrightarrow{\nabla}\cdot\overrightarrow{B} = 0, \quad \overrightarrow{\nabla}\times\overrightarrow{B} = \frac{\partial\overrightarrow{E}}{\partial t} + \overrightarrow{J}.\end{aligned}$$

The second and fourth equations are solved by

$$\overrightarrow{E} = -\overrightarrow{\nabla} A^0 - \frac{\partial \overrightarrow{A}}{\partial t}, \quad \overrightarrow{B} = \overrightarrow{\nabla} \times \overrightarrow{A},$$

and are invariant under the gauge transformations

$$\overrightarrow{A}' = \overrightarrow{A} - \overrightarrow{\nabla}\Lambda, \quad A_0' = A_0 + \frac{\partial \Lambda}{\partial t}.$$

Therefore one can impose a gauge choice, the Lorentz gauge, on the potential $\overrightarrow{A}$ and A_0

$$\nabla \cdot A + \frac{\partial A^0}{\partial t} = 0.$$

In this gauge, Maxwell equations simplify to

$$\eta^{\mu\nu}\partial_\mu\partial_\nu A^0 = -\rho, \quad \eta^{\mu\nu}\partial_\mu\partial_\nu \overrightarrow{A} = \overrightarrow{J},$$

which is a standard wave equation with a source. Maxwell equations could be written in a Lorentz covariant manner if we define the curvature

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$$

so that

$$\partial_\nu F^{\mu\nu} = J^\nu.$$

We can replace the definition of $F_{\mu\nu}$ by the requirement

$$\partial_\rho F_{\mu\nu} + \partial_\mu F_{\nu\rho} + \partial_\nu F_{\rho\mu} = 0,$$

which, locally, can be solved by the above equation defining A_μ. The curvature $F_{\mu\nu}$ is invariant under the gauge transformations $A_\mu' = A_\mu + \partial_\mu \Lambda$. Maxwell equations could be obtained from the action

$$I_1 = \int d^4x \left(-\frac{1}{4} F_{\mu\nu} F_{\rho\sigma} \eta^{\mu\rho} \eta^{\nu\sigma} + A_\mu J^\mu \right).$$

According to Noether's theorem, there is a conserved charged associated to every symmetry of the action. The symmetry in this case gives electric charge conservation. To find the interactions of electrons with photons, we note that the action for electrons $I_{\frac{1}{2}}$ is invariant under the global symmetry $\psi(x) \to e^{-ie\Lambda}\psi(x)$ where Λ is a constant phase. This global symmetry is lost if the symmetry is promoted to a local symmetry by allowing Λ to depend on x. In this case

$$\overline{\psi}(x) i\gamma^\mu \partial_\mu \psi(x) \to \overline{\psi}(x) i\gamma^\mu \partial_\mu \psi(x) + e\partial_\mu \Lambda \overline{\psi}(x) \gamma^\mu \psi(x).$$

To restore the symmetry, we add to the Dirac-action, the photon-electron interaction term

$$-e \int d^4x \overline{\psi}(x) \gamma^\mu \psi(x) A_\mu,$$

which transforms under the local symmetry according to

$$-e\overline{\psi}(x)\gamma^{\mu}\psi(x)A_{\mu} \to -e\overline{\psi}(x)\gamma^{\mu}\psi(x)A_{\mu} - e\partial_{\mu}\Lambda\overline{\psi}(x)\gamma^{\mu}\psi(x).$$

We deduce that the action for the spin-$\frac{1}{2}$, spin-1 system governing the interaction of photons and electrons and invariant under the local gauge transformations $\psi(x) \to e^{-ie\Lambda}\psi(x)$ and $A_{\mu} \to A_{\mu} + \partial_{\mu}\Lambda$, is given by

$$I_{\left(\frac{1}{2},1\right)} = \int d^4x \left(-\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \overline{\psi}(x)\left(i\gamma^{\mu}D_{\mu} - m\right)\psi(x) \right),$$

where $D_{\mu} = \partial_{\mu} + ieA_{\mu}$. Therefore, interaction terms are obtained by replacing ordinary derivatives ∂_{μ} in the free Dirac action by covariant derivatives D_{μ}. These covariant derivatives satisfy the commutation relations

$$[D_{\mu}, D_{\nu}] = ieF_{\mu\nu}.$$

4.4 Spin-$\frac{3}{2}$: Rarita-Schwinger field

The field that describes a spin-$\frac{3}{2}$ particle is a vector-spinor $\psi_{\mu\alpha}$ where μ is a space-time vector index and α is a spinor index. The free field satisfies the Rarita-Schwinger equation

$$\gamma^{\mu\nu\rho}\psi_{\nu\rho} = 0$$

where $\psi_{\mu\nu} = \partial_{\mu}\psi_{\nu} - \partial_{\nu}\psi_{\mu}$, and

$$\gamma^{\mu\nu\rho} = \frac{1}{3!}\left(\gamma^{\mu}\gamma^{\nu}\gamma^{\rho} - \gamma^{\nu}\gamma^{\mu}\gamma^{\rho} + \gamma^{\nu}\gamma^{\rho}\gamma^{\mu} - \gamma^{\rho}\gamma^{\nu}\gamma^{\mu} + \gamma^{\rho}\gamma^{\mu}\gamma^{\nu} - \gamma^{\mu}\gamma^{\rho}\gamma^{\nu}\right)$$

is completely antisymmetric in the indices $\mu\nu\rho$. The product of spin-1 and spin-$\frac{1}{2}$ representations gives spin-$\frac{3}{2}$ and spin-$\frac{1}{2}$ representations

$$\left(\frac{1}{2}, \frac{1}{2}\right) \otimes \left(\frac{1}{2}, 0\right) = \left(1, \frac{1}{2}\right) \oplus \left(0, \frac{1}{2}\right).$$

It will be seen that the Rarita-Schwinger equation is necessary to eliminate the spin-$\frac{1}{2}$ component from $\psi_{\mu\alpha}$.

4.5 Spin-2 : Gravitational field

The massless spin-2 field is the graviton which is a symmetric traceless tensor $h_{\mu\nu}$ with gauge invariance

$$h_{\mu\nu} \to h_{\mu\nu} + \partial_{\mu}\xi_{\nu} + \partial_{\nu}\xi_{\mu}.$$

To see that this is none other than the linearized form of the metric tensor of the underlying space-time manifold, let us assume that we would like to promote the global Lorentz invariance of spinors to a local one:

$$\psi_{\alpha}(x) \to \exp\left(\frac{i}{4}\Lambda^{ab}(x)\gamma_{ab}\right)_{\alpha}^{\beta}\psi_{\beta}(x),$$

where $\gamma_{ab} = \frac{1}{2}\left(\gamma^a\gamma^b - \gamma^b\gamma^a\right)$. To restore the invariance of the Dirac equation under local Lorentz transformations, and in analogy with the electromagnetic case, we introduce the covariant derivative

$$D_\mu = \partial_\mu + \frac{1}{4}\omega_\mu{}^{ab}\gamma_{ab}.$$

The term $\overline{\psi}(x)\left(i\gamma^\mu D_\mu - m\right)\psi(x)$ becomes invariant provided that

$$\omega'^{ab}_\mu = \partial_\mu\Lambda^{ab} + \omega_\mu{}^{ac}\Lambda_c^{\ b} - \omega_\mu{}^{bc}\Lambda_c^{\ a}.$$

The curvature tensor is defined by

$$\left[D_\mu, D_\nu\right] = \frac{1}{4}R_{\mu\nu}{}^{ab}\gamma_{ab},$$

which can be easily evaluated to be

$$R_{\mu\nu}{}^{ab} = \partial_\mu\omega_\nu{}^{ab} - \partial_\nu\omega_\mu{}^{ab} + \omega_\mu{}^{ac}\omega_{\nu c}{}^{b} - \omega_\nu{}^{ac}\omega_{\mu c}{}^{b},$$

and is covariant under local Lorentz transformations. To make contact with the Riemann curvature tensor, we introduce the soldering form e_μ^a satisfying

$$\nabla_\mu e_\nu^a = \partial_\mu e_\nu^a - \Gamma_{\mu\nu}^\rho e_\rho^a + \omega_\mu{}^{ab}e_{\nu b} = 0,$$

where $\Gamma_{\mu\nu}^\rho = \frac{1}{2}g^{\rho\sigma}\left(g_{\mu\sigma,\nu} + g_{\nu\sigma,\mu} - g_{\mu\nu,\sigma}\right)$ is the Christoffel connection and $g_{\mu\nu} = e_\mu^a\eta_{ab}e_\nu^b$. This equation implies that the covariant derivative of the metric under coordinate transformations vanishes. The constraint equation can be solved for $\omega_\mu{}^{ab}$ in terms of e_μ^a. Substituting the solution into $R_{\mu\nu}{}^{ab}(\omega)$ one can show that

$$R_{\mu\nu}{}^{ab}(\omega)\, e_{\rho a}e_{\sigma b} = R_{\mu\nu\rho\sigma}(g),$$

is identical to the Riemann tensor as a function of the metric g.

Equivalently, we can introduce gauge fields corresponding to both the Lorentz generators J_{ab} and translations P_a

$$D_\mu = \partial_\mu + \omega_\mu{}^{ab}J_{ab} + e_\mu^a P_a,$$

then the curvature is

$$\left[D_\mu, D_\nu\right] = R_{\mu\nu}{}^{ab}J_{ab} + T_{\mu\nu}^a P_a,$$

where $R_{\mu\nu}{}^{ab}$ is the same as before while

$$T_{\mu\nu}^a = \partial_\mu e_\nu^a - \partial_\nu e_\mu^a + \omega_\mu{}^{ab}e_{\nu b} - \omega_\nu{}^{ab}e_{\mu b}.$$

By setting the torsion $T_{\mu\nu}^a$to zero we can uniquely solve for $\omega_\mu{}^{ab}$ in terms of e_μ^a and substitute this expression into $R_{\mu\nu}{}^{ab}$ to obtain the Riemann tensor.

This last formulation is related to the Cartan structure equations

$$T^a = de^a + \omega^{ab} \wedge e_b,$$
$$R^{ab} = d\omega^{ab} + \omega^{ac} \wedge \omega_c^{\ b}.$$

By writing $e^a = e_\mu^a dx^\mu$, $\omega^{ab} = \omega_\mu^{ab} dx^\mu$, $R^{ab} = \frac{1}{2} R_{\mu\nu}{}^{ab} dx^\mu \wedge dx^\nu$ and $T^a = \frac{1}{2} T_{\mu\nu}^a dx^\mu \wedge dx^\nu$ we can recover all the previous formulas.

The Einstein-Hilbert action is given by

$$I_2 = -\int d^4x \frac{e}{4} e_a^\mu e_b^\nu R_{\mu\nu}{}^{ab},$$

where $e = \det \left(e_\mu^a\right)$. The dynamical equations governing the interaction of electrons and photons with gravity can be read from the equation

$$I_{\left(\frac{1}{2},1,2\right)} = \int d^4xe \left(\overline{\psi} \left(i\gamma^\mu \left(\partial_\mu + eA_\mu + \frac{1}{4} \omega_\mu^{\ ab} \gamma_{ab} \right) - m \right) \psi \right)$$
$$- \int d^4xe \left(\frac{1}{4} F_{\mu\nu} F_{\rho\sigma} g^{\mu\rho} g^{\nu\sigma} + \frac{1}{4} e_a^\mu e_b^\nu R_{\mu\nu}{}^{ab} \right).$$

4.6 The standard model

The principle of gauge invariance plays a fundamental role in physics. The Poincare symmetry is a space-time symmetry effecting local space-time coordinates. On the other hand the known elementary particles in nature, fall into group representations, e.g. fermions have the local gauge invariance under the transformations $\psi_{\alpha I} \rightarrow \left(e^{ig_a \Lambda^a(x) T^a}\right)_I^J \psi_{\alpha J}$ where $(T^a)_I^J$ are the representations of the gauge group. As in the Maxwell case, connections are introduced to insure local gauge invariance. Let

$$D = d + igA, \quad A = A_\mu^a T^a dx^\mu,$$

so that the curvature of the connection A will be given by

$$D^2 = igF = ig\left(dA + igA^2\right),$$
$$F = \frac{1}{2} F_{\mu\nu}^a T^a dx^\mu \wedge dx^\nu,$$

so that

$$F_{\mu\nu}^a = \partial_\mu A_\nu^a - \partial_\nu A_\mu^a + g f^{abc} A_\mu^b A_\nu^c,$$

where $\left[T^a, T^b\right] = i f^{abc} T^c$. Let us denote the quarks by $q = \begin{pmatrix} u_L \\ d_L \end{pmatrix}$, u_R, d_R which are in the representations $\left(3, 2, -\frac{1}{3}\right)$, $\left(\overline{3}, 1, -\frac{4}{3}\right)$ and $\left(\overline{3}, 1, \frac{2}{3}\right)$ of $SU(3)_c \times SU(2)_w \times U(1)_Y$. The leptons are denoted by $l = \begin{pmatrix} e_L^- \\ \nu_L \end{pmatrix}$, e_R^+ which are in the

$(1,2,1)$ and $(1,1,2)$ of $SU(3)_c \times SU(2)_w \times U(1)_Y$.. The Lagrangian that govern all known interactions is given by

$$\begin{aligned}
e^{-1}L &= -\frac{1}{4\kappa^2}R - \frac{1}{4}\left(F_{\mu\nu}^i F^{\mu\nu i} + F_{\mu\nu}^\alpha F^{\mu\nu\alpha} + B_{\mu\nu}B^{\mu\nu}\right) \\
&+ D_\mu H^\dagger D^\mu H - \mu^2 H^\dagger H + \lambda\left(H^\dagger H\right)^2 \\
&+ i\overline{q}\gamma^\mu\left(\partial_\mu + \frac{1}{4}\omega_\mu^{\ ab}\gamma_{ab} - \frac{i}{2}g_2 A_\mu^\alpha\sigma^\alpha - \frac{i}{6}g_1 B_\mu - \frac{i}{2}g_3 V_\mu^i\lambda^i\right)q \\
&+ i\overline{d}_R\gamma^\mu\left(\partial_\mu + \frac{1}{4}\omega_\mu^{\ ab}\gamma_{ab} + \frac{i}{3}g_1 B_\mu - \frac{i}{2}g_3 V_\mu^i\lambda^i\right)d_R \\
&+ i\overline{u}_R\gamma^\mu\left(\partial_\mu + \frac{1}{4}\omega_\mu^{\ ab}\gamma_{ab} - \frac{2i}{3}g_1 B_\mu - \frac{i}{2}g_3 V_\mu^i\lambda^i\right)u_R \\
&+ i\overline{l}\gamma^\mu\left(\partial_\mu + \frac{1}{4}\omega_\mu^{\ ab}\gamma_{ab} - \frac{i}{2}g_2 A_\mu^\alpha\sigma^\alpha + \frac{i}{2}g_1 B_\mu\right)l \\
&+ i\overline{e}_R\gamma^\mu\left(\partial_\mu + \frac{1}{4}\omega_\mu^{\ ab}\gamma_{ab} + ig_1 B_\mu\right)e_R \\
&+ \left(k^d\overline{q}Hd_R + k^u\overline{q}\tau^2 Hu_R + k^e\overline{l}He_R + h.c.\right),
\end{aligned}$$

where the Higgs field H is in the $(1,2,1)$ representation so that $D_\mu H = \partial_\mu H - \frac{i}{2}g_2 A_\mu^\alpha\sigma^\alpha - \frac{i}{2}g_1 B_\mu H$ and V_μ^i, A_μ^α and B_μ are the gauge fields for the gauge Lie algebras $SU(3)_c$, $SU(2)_w$ and $U(1)_Y$. The k^d, k^u and k^e are 3×3 matrices mixing the three generations of particles. Minimizing the potential of the Higgs fields H gives a vacuum

$$\langle H\rangle = \begin{pmatrix} 0 \\ v \end{pmatrix},$$

which generates masses for the quarks and leptons, e.g. the leptons will acquire the term

$$k^e\overline{l}He_R = k^e v\overline{e}_L e_R.$$

Similarly the term $D_\mu H^\dagger D^\mu H$ will yield the mass terms

$$\frac{1}{4}v^2 g_2^2\left(W_\mu^+ W^{\mu-} + \frac{1}{\cos^2\theta}Z_\mu Z^\mu\right),$$

where $W_\mu^\pm = \left(A_\mu \pm iA_\mu^2\right)$, $Z_\mu = \frac{1}{\sqrt{g_1^2+g_2^2}}\left(g_2 A_\mu^3 - g_1 B_\mu\right)$ and $\sin^2\theta = \frac{g_1^2}{g_1^2+g_2^2}$. The symmetry of the Lagrangian is diffeomorphism$\times$ local internal symmetry. All fermions are chiral (Weyl fermions) and acquire mass only after the symmetry is broken spontaneously from $SU(3)_c\times SU(2)_w\times U(1)_Y$ to $SU(3)_c\times U(1)_{em}$. At the quantum level chiral fermions break gauge invariance. The chirality condition is $\gamma_5\psi_\pm = \pm\psi_\pm$ where $\gamma_5 = i\gamma_0\gamma_1\gamma_2\gamma_3$ so that $\gamma_5^2 = 1$. A Dirac spinor is decomposed according to $\psi = \psi_+ + \psi_-$. The eigenvalue equation $D\psi = \lambda\psi$ implies $\gamma_5 D\psi_\pm = \mp\lambda D\psi_\pm$ so that the eigenvalue problem could not be set for chiral fermions alone without their partners except for zero eigevalues. After quantization one must sum over all states which would involve computing the determinat

$$\int D\psi D\overline{\psi} e^{\overline{\psi} D\psi} \to \det D,$$

and this involve the step of taking the trace over all states, which must now be restricted to the chiral ones

$$\langle \psi_{n-} |\, D\, | \psi_{n+} \rangle .$$

This is not invariant under the chiral rotation $\psi_{n+} \to e^{i\theta\gamma_5}\psi_{n+}$. To see this explicitely we have

$$Tr(F(D)) = \sum_n \langle \psi_{n-} |\, F(D)\, | \psi_{n+} \rangle \to \sum_n \left\langle e^{-i\theta^a T^a} \psi_{n-} \right| F(D) \left| e^{i\theta^a T^a} \psi_{n+} \right\rangle .$$

Since $F(D)$ depends only on geometrical quantities, one can use the heat kernel expansion to find the lowest terms

$$\begin{aligned} Tr(F(D)) &\to 2i\theta^a Tr\left(\gamma_5 T^a \gamma^{\mu\nu} F^b_{\mu\nu} T^b \gamma^{\rho\sigma} F^c_{\rho\sigma} T^c\right) \\ &= 2i\theta^a \epsilon^{\mu\nu\rho\sigma} F^b_{\mu\nu} F^c_{\rho\sigma} Tr\left(T^a \left\{T^b, T^c\right\}\right) . \end{aligned}$$

We deduce that gauge invariance is not destroyed by chirality if

$$Tr\left(T^a \left\{T^b, T^c\right\}\right) = 0.$$

If we include gravitational terms then the lowest order term is

$$2i\theta^a Tr\left(\gamma_5 T^a \gamma^{\mu\nu} R_{\mu\nu}{}^{cd} \gamma^{\rho\sigma} R_{\rho\sigma cd}\right),$$

which vanishes if

$$Tr\left(T^a\right) = 0,$$

insuring the absence of gravitational anomalies. Both conditions on the representations of the chiral fermions in the standard model are satisfied. These are very strong constraints on possible representations of the fermions in a realistic model and lead to the conclusion that these are not satisfied by accident but are derivable from higher principles. One such explanation is unification where the chiral fermionic representations

$$\left(3, 2, -\frac{1}{3}\right) \oplus \left(\overline{3}, 1, \frac{2}{3}\right) \oplus \left(\overline{3}, 1, -\frac{4}{3}\right) \oplus (1, 2, 1) \oplus (1, 1, 2)$$

could be obtained from the $\overline{5} \oplus 10$ representations of $SU(5)$ to be broken spontaneously into two stages

$$SU(5) \to SU(3)_c \times SU(2)_w \times U(1)_Y \to SU(3)_c \times U(1)_{em}.$$

This is not the unique possibility but the simplest.

5 Supersymmetry

In nature there are particles of different spins, in particular integral and half-integral spins. In the ultimate theory there must be a symmetry that places particles with different spin in the same multiplet. Until 1974 all internal symmetries were studied and there was a theorem by Coleman and Mandula stating that, under certain physical assumptions, it is impossible to have a space-time symmetry larger than the Poincare symmetry. This obstruction was overcome by the simple extension of considering $\mathbb{Z}_2$ graded algebras whose generators are classified into two classes, even (bosonic) and odd (fermionic) and obey

$$[even, even] = even, \quad [even, odd] = odd, \quad \{odd, odd\} = even.$$

With every generator A in the graded Lie algebra we associate a number $a = \begin{cases} 0 \text{ if } A \text{ is even} \\ 1 \text{ if } A \text{ is odd} \end{cases}$ and define the graded commutator by

$$[A, B\} = AB - (-1)^{ab} BA,$$

which satisfies the graded Jacobi identity

$$[A, [B, C\}\} + (-1)^{a(b+c)} [B, [C, A\}\} + (-1)^{c(a+b)} [C, [A, B\}\} = 0.$$

Denote by Q^i_α the fermionic generators where the index α transforms under the Lorentz group and the index i transforms under an internal algebra with generators T_r

$$\begin{aligned}
[T_r, T_s] &= f_{rst} T_t, \quad [P_\mu, T_r] = 0, \quad [J_{\mu\nu}, T_r] = 0, \\
\left[Q^i_\alpha, J_{\mu\nu}\right] &= (b_{\mu\nu})_\alpha^{\ \beta} Q^i_\beta, \quad \left[Q^i_\alpha, P_\mu\right] = (c_\mu)_\alpha^{\ \beta} Q^i_\beta, \\
\left\{Q^i_\alpha, Q^j_\beta\right\} &= r\left(\gamma^\mu C\right)_{\alpha\beta} P_\mu \delta^{ij} + s\left(\gamma^{\mu\nu} C\right)_{\alpha\beta} J_{\mu\nu} \delta^{ij} \\
&\quad + C_{\alpha\beta} U^{ij} + (\gamma_5 C)_{\alpha\beta} V^{ij} + (\gamma^\mu \gamma_5 C)_{\alpha\beta} L^{ij}_\mu.
\end{aligned}$$

By the Coleman-Mandula theorem we can set L^{ij}_μ to zero because it carries a Lorentz vector index. Without any loss in generality we can assume that Q^i_α satisfy the Majorana condition $Q^i_\alpha = C_{\alpha\beta} \overline{Q}^{\beta i}$.

Using the Jacobi identity

$$\left[J_{\mu\nu}, \left[J_{\rho\sigma}, Q^i_\alpha\right]\right] + \left[J_{\rho\sigma}, \left[Q^i_\alpha\right]\right] + \left[Q^i_\alpha, \left[J_{\mu\nu}, J_{\rho\sigma}\right]\right] = 0,$$

we deduce that $(b_{\mu\nu})_\alpha^{\ \beta}$ should form a representation of the Lorentz group and must be identified with

$$(b_{\mu\nu})_\alpha^{\ \beta} = \frac{i}{2} (\gamma_{\mu\nu})_\alpha^{\ \beta}$$

From the identity

$$\left[P_\mu, \left[P_\nu, Q^i_\alpha\right]\right] + \left[P_\nu, \left[Q^i_\alpha, P_\mu\right]\right] + \left[Q^i_\alpha, \left[P_\mu, P_\nu\right]\right] = 0,$$

one deduces that $[c_\mu, c_\nu] = 0$. But it is always possible to change $(c_\mu)_\alpha^\beta$ to the form $c_\mu = c\,(\gamma_\mu)_\alpha^\beta$ and this implies that $c = 0$. From the identity

$$\left[Q_\alpha^i, \left\{Q_\beta^j, Q_\gamma^k\right\}\right] + \left[Q_\beta^j, \{Q_\gamma^k, Q_\alpha^i\}\right] + \left[Q_\gamma^k, \left\{Q_\alpha^i, Q_\beta^j\right\}\right] = 0,$$

we deduce that U^{ij} and V^{ij} commute with all the generators. Finally from the identity

$$\left[P_\mu, \left\{Q_\alpha^i, Q_\beta^j\right\}\right] + \left\{Q_\alpha^i, \left[Q_\beta^j, P_\mu\right]\right\} - \left\{Q_\beta^j, \left[P_\mu, Q_\alpha^i\right]\right\} = 0,$$

we deduce that $s = 0$. We shall rescale the parameter r to 2. Summarizing the results we have the most general graded algebra which is an extension of the Poincare symmetery and consistent with unitarity:

$$\begin{aligned} \left[Q_\alpha^i, J_{\mu\nu}\right] &= -\frac{i}{2}\,(\gamma_{\mu\nu})_\alpha^\beta\, Q_\beta^i, \quad \left[Q_\alpha^i, P_\mu\right] = 0, \\ \left\{Q_\alpha^i, Q_\beta^j\right\} &= -2\,(\gamma^\mu C)_{\alpha\beta}\, P_\mu \delta^{ij} + C_{\alpha\beta} U^{ij} + (\gamma_5 C)_{\alpha\beta}\, V^{ij}. \end{aligned}$$

5.1 Irreducible representations of supersymmetry

The Casimir operators are P^2 and $\widehat{W}^2$ where

$$\widehat{W}_\mu = \epsilon_{\mu\nu\rho\sigma} P^\nu J^{\rho\sigma} + \overline{Q}^i \gamma_\mu \gamma_5 Q^i.$$

Since P^2 is a Casimir, all particles belonging to the same multiplet must have the same mass. Now introduce the fermion number $(-1)^{N_F}$ which acts on bosonic and fermionic states as follows

$$\begin{aligned} (-1)^{N_F} \left|boson\right\rangle &= \left|boson\right\rangle, \\ (-1)^{N_F} \left|fermion\right\rangle &= -\left|fermion\right\rangle, \end{aligned}$$

which implies that

$$(-1)^{N_F}\, Q_\alpha^i \left|.\right\rangle = -Q_\alpha^i\, (-1)^{N_F} \left|.\right\rangle.$$

For any finite dimensional representation we have

$$\begin{aligned} Tr\left((-1)^{N_F}\left\{Q_\alpha^i, \overline{Q}^{i\beta}\right\}\right) &= Tr\left((-1)^{N_F}\,(Q_\alpha^i \overline{Q}^{i\beta} + \overline{Q}^{i\beta} Q_\alpha^i\right) \\ &= Tr\left((-1)^{N_F}\,(\gamma^\mu)_\alpha^\beta\, P_\mu\right) \\ &= Tr\left(-Q_\alpha^i\,(-1)^{N_F}\,\overline{Q}^{i\beta} + Q_\alpha^i\,(-1)^{N_F}\,\overline{Q}^{i\beta}\right) = 0. \end{aligned}$$

From this we deduce that

$$Tr\left((-1)^{N_F}\right) = 0,$$

implying that the number of fermionic and bosonic degrees of freedom are identical.

The energy in supersymmetric theories is non-negative. To see this consider

$$0 \leq \sum_i \left(Q^i_\alpha \left(Q^i_\alpha\right)^\dagger + \left(Q^i_\alpha\right)^\dagger Q^i_\alpha \right) = \sum_i \left(Q^i_\alpha \left(\overline{Q}^i \gamma_0\right)^\alpha + \left(\overline{Q}^i \gamma_0\right)^\alpha Q^i_\alpha \right)$$
$$= Tr \sum_i \left\{ Q^i_\alpha, \left(\overline{Q}^i \gamma_0\right)^\alpha \right\} = 2NTr\left(\gamma^\mu P_\mu \gamma_0\right) = 8NP_0.$$

Therefore $E \geq 0$.

We would like to characterize the massless representations of supersymmetry: $P^2 = 0$, and we can choose $P_\mu = (w, 0, 0, w)$. In this frame, and after setting the central charges U^{ij} and V^{ij} to zero, we get

$$\left\{ Q^i_\alpha, Q^j_\beta \right\} = 2\delta^{ij} \left(\gamma^\mu C\right)_{\alpha\beta} P_\mu = 2w\delta^{ij} \left(\left(\gamma_0 + \gamma_3\right) C\right)$$
$$= -4w\delta^{ij} \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}.$$

We therefore have one independent relation $\left\{ Q^i_1, Q^j_4 \right\} = -4E\delta^{ij}$. From the relation $Q^i_\alpha = C_{\alpha\beta} \overline{Q}^{i\beta}$ we have

$$Q^i_3 = Q^{i\dagger}_2, \quad Q^i_4 = -Q^{i\dagger}_1$$

so we can write $\left\{ Q^i_1, Q^{j\dagger}_1 \right\} = 4w\delta^{ij}$. Normalizing the Q^i_1 we can finally write

$$\left\{ Q^i_1, Q^{j\dagger}_1 \right\} = \delta^{ij}.$$

The subgroup of the Lorentz group leaving the state $(w, 0, 0, w)$ invariant is $\Lambda_{03} = 0$, $\Lambda_{10} = -\Lambda_{13}$, $\Lambda_{20} = -\Lambda_{23}$ implying that the generators are

$$T_1 = J_{10} + J_{13}, \quad T_2 = J_{20} + J_{23}, \quad J = J_{12}.$$

These obey the commutation relations

$$[J, T_1] = T_2, \quad [J, T_2] = -T_1, \quad [T_1, T_2] = 0,$$
$$\left[J, Q^i_1\right] = \frac{1}{2} Q^i_1, \quad \left[J, \overline{Q}^i_1\right] = -\frac{1}{2} \overline{Q}^i_1, \quad \left\{ Q^i_1, Q^{j\dagger}_1 \right\} = \delta^{ij}.$$

Since Q^i_1 and $Q^{i\dagger}_1$ form a Clifford algebra, they correspond to a set of N fermi creation operators $Q^{i\dagger}_1$ and N fermi annihilation operators Q^i_1. The vacuum state is defined by $|\lambda\rangle$

$$Q^i_1 |\lambda\rangle = 0, \quad J|\lambda\rangle = \lambda |\lambda\rangle,$$
$$Q^{i\dagger}_1 |\lambda\rangle = \left|\lambda - \frac{1}{2}, i\right\rangle, \quad Q^{i\dagger}_1 Q^{j\dagger}_1 |\lambda\rangle = |\lambda - 1, [ij]\rangle,$$
$$Q^{1\dagger}_1 Q^{2\dagger}_1 \cdots Q^{N\dagger}_1 |\lambda\rangle = \left|\lambda - \frac{N}{2}\right\rangle.$$

These form 2^N states falling into two classes. Each class has 2^{N-1} states obtained from $|\lambda\rangle$ by acting with odd and even number of $Q_1^{i\dagger}$ respectively. The 2^N states have helicities ranging from $\lambda_{\rm max}$ to $\lambda_{\rm max}-\frac{N}{2}$. By the CPT theorem physical states must have both helicities λ and $-\lambda$ present which implies that $\lambda_{\rm max}-\frac{N}{2}=-\lambda_{\rm max}$ and therefore $\lambda_{\rm max}=\frac{N}{4}$. If this is not satisfied one must add the conjugate states where $\lambda'_{\rm max}=\frac{N}{2}-\lambda_{\rm max}$ and act on this state with $Q_1^{i\dagger}$. A massless non CPT conjugate multiplet has 2^{N+1} states. For gauge and matter theories $\lambda_{\rm max}\leq 1$ which implies that $N\leq 4$. We can construct the following table:

	I	II	III	IV	V	VI
λ	$N=1$	$N=2$	$N=1$	$N=2$	$N=3$	$N=4$
1		1		1	1	1
$\frac{1}{2}$	1	1	$1+1'$	2	$3+1$	4
0	$1+1$		$2+2'$	$1+1'$	$\overline{3}+3$	6
$-\frac{1}{2}$	1	$1'$	$1+1'$	2	$1+\overline{3}$	$\overline{4}$
-1		1		1	1	1

The multiplets in cases I and II have $\lambda_{\rm max}=\frac{1}{2}$, while in all the other cases $\lambda_{\rm max}=1$. Case I corresponds to an $N=1$ chiral multiplet with $2^{1+1}=4$ states. Case II is $N=1$ vector multiplet also with $2^{1+1}=4$ states. Case III is an $N=2$ vector multiplet with $2^{2+1}=8$ states. Case IV is an $N=2$ hypermultiplet with $2^{2+1}=8$ states. Finally case VI give the $N=4$ vector multiplet with $2^4=16$ states as it is CPT self conjugate. The $N=3$ in case V is identical to $N=4$ because one must add the CPT conjugate states.

Next we consider the situation where $\lambda_{\rm max}=2$ which implies that $N\leq 8$. All these multiplets will contain the graviton and will correspond to supergravity multiplets:

λ	$N=1$	$N=2$	$N=3$	$N=4$	$N=5$	$N=6$	$N=7$	$N=8$
2	1	1	1	1	1	1	1	1
$\frac{3}{2}$	1	2	3	4	5	6	$7+1$	8
1		1	$\overline{3}$	6	$10+1$	15	$21+7$	28
$\frac{1}{2}$			1	$\overline{4}$	$\overline{10}+1$	$20+1$	$35+21$	56
0				$1+1$	$\overline{5}+5$	$\overline{6}+6$	$\overline{35}+35$	70
$-\frac{1}{2}$			1	4	$1+10$	$1+\overline{20}$	$\overline{21}+\overline{35}$	$\overline{56}$
-1		1	3	6	$\overline{10}+1$	$\overline{15}$	$\overline{21}+\overline{7}$	$\overline{28}$
$-\frac{3}{2}$	1	2	$\overline{3}$	$\overline{4}$	$\overline{5}$	$\overline{6}$	$\overline{7}+1$	$\overline{8}$
-2	1	1	1	1	1	1	1	1

Notice that $N=7$ supergravity is identical to $N=8$ supergravity, so there are seven different supergravity theories. Only the field content of the first three supergravity theories is fixed uniquely. The simplest is $N=1$ supergravity where only two fields are needed, the metric (graviton) and the gravitino (Rarita-Schwinger).

5.2 Supersymmetric field theories

5.2.1 Lagrangian for chiral multiplet

The Lagrangian for $N = 1$ chiral multiplet (A, B, ψ) is given by

$$L = \frac{1}{2}\left(\partial_\mu A \partial^\mu A + \partial_\mu B \partial^\mu B + i\overline{\psi}\gamma^\mu \partial_\mu \psi\right).$$

The action is invariant under the supersymmetry transformations

$$\begin{aligned} &\delta A = \overline{\epsilon}\psi, \quad \delta B = \overline{\epsilon}\gamma_5\psi, \\ &\delta\psi = \left(-i\partial_\mu A + \partial_\mu B\gamma_5\right)\gamma^\mu\epsilon, \end{aligned}$$

as can be easily verified. The supersymmetry parameter ϵ is independent of the coordinates x^μ and supersymmetry is a global symmetry.

5.2.2 Lagrangian for vector multiplet

The vecor multiplet is given by (A^μ, λ) where $A_\mu = A_\mu^a T^a$ and $\lambda = \lambda^a T^a$ which are Lie algebra valued: $\left[T^a, T^b\right] = if^{abc}T^c$. The Lagrangian which is valid in dimensions $D = 4, 6$ and 10 is given by

$$L = Tr\left(-\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{i}{2}\lambda\Gamma^\mu D_\mu\lambda\right),$$

where $F = dA + A^2 = \frac{1}{2}F_{\mu\nu}dx^\mu \wedge dx^\nu$, $F_{\mu\nu} = F_{\mu\nu}^a T^a$ and $D_\mu\lambda = \partial_\mu\lambda + ig\left[A_\mu, \lambda\right] = D_\mu\lambda^a T^a$. The action is invariant under the supersymmetry transformations

$$\begin{aligned} \delta A_\mu &= i\epsilon\Gamma_\mu\lambda, \\ \delta\lambda &= \frac{1}{2}F_{\mu\nu}\Gamma^{\mu\nu}\epsilon. \end{aligned}$$

To verify supersymmetry one has to use Fierz identities for the fermions which are valid only in dimensions $4, 6$ and 10.

One can couple vector multiplets to chiral multiplets and have a generalization of the standard model. In the supersymmetric standard model it turned out that all known fields must have partners, e.g., the electron must have a bosonic partner, the s-electron, while gauge fields A_μ will have as partners the gauginos λ. As we have seen, the fermions in the standard model are chiral. The chirality condition cannot be satisfied for theories with more than $N = 1$ supersymmetry. In other words, for supersymmetry to be of any relevance, the Lagrangian at low energies must have $N = 1$ supersymmetry. However, in the real world supersymmetry is not present because no partners for the observed particles with the same mass are found, so it must be broken. One of the main advantages of supersymmetry is that it has good quantum behaviour, where divergencies in the renormalization of the bosonic fields are cancelled by those coming from the fermions. To preserve good quantum behaviour, the symmetry must be broken spontaneously. The Higgs mechanism which is essential in the bosonic case must also be present here. The Goldstone phenomena where the scalar fields associated with the broken generators are absorbed by the massless vector fields to become massive must be generalized. In this case the field associated with the broken supersymmetry generator, the Goldstino, must be absorbed by a massless vector-spinor field. In the discussion in the last section we have seen that this is the Rartia-Schwinger field. This implies that the supergravity multiplet must be coupled to the matter Lagrangian of the standard model in such a way that supersymmetry is broken spontaneously, so that the gravitino absorb one fermionic field to become massive. In supergravity the gravitational multiplet plays a role in the breaking of supersymmetry and this can be used to induce the electroweak breaking in the standard model. This is a novel phenomena as this implies that physics of the extremely weak gravitational field influences the low-energy sector.

5.2.3 N=1 supergravity Lagrangian

The easiest way to construct the supergravity Lagrangian is to generalize the construction of the Einstein-Hilbert action based on gauging the Poincare algebra. Here we gauge the supersymmetry algebra instead. Let

$$D_\mu = \partial_\mu + \omega_\mu{}^{ab} J_{ab} + e_\mu^a P_a + \psi_\mu^\alpha Q_\alpha,$$

be the connection associated with the supersymmetry algebra. The curvature is easily computed to be

$$[D_\mu, D_\nu] = R_{\mu\nu}{}^{ab} J_{ab} + T_{\mu\nu}^a P_a + \psi_{\mu\nu}^\alpha Q_\alpha,$$

where

$$\begin{aligned} R_{\mu\nu}{}^{ab} &= \partial_\mu \omega_\nu{}^{ab} - \partial_\nu \omega_\mu{}^{ab} + \omega_\mu{}^{ac} \omega_{\nu c}{}^{b} - \omega_\nu{}^{ac} \omega_{\mu c}{}^{b}, \\ T_{\mu\nu}^a &= \partial_\mu e_\nu^a - \partial_\nu e_\mu^a + \omega_\mu{}^{ab} e_{\nu b} - \omega_\nu{}^{ab} e_{\mu b} + \overline{\psi}_\mu \gamma^a \psi_\nu, \\ \psi_{\mu\nu} &= \left(\partial_\mu + \frac{1}{4}\omega_\mu{}^{ab}\gamma_{ab}\right)\psi_\nu - \left(\partial_\nu + \frac{1}{4}\omega_\nu{}^{ab}\gamma_{ab}\right)\psi_\mu, \end{aligned}$$

and the action of J_{ab} on the spinor ψ_μ is represented by $\frac{1}{4}\gamma_{ab}$. To proceed one first impose the torsion free constraint

$$T_{\mu\nu}^a = 0,$$

which can be solved to express $\omega_\mu{}^{ab}$ in terms of e_μ^a and ψ_μ to obtain

$$\omega_\mu{}^{ab} = \omega_\mu{}^{ab}(e) + \frac{1}{4}\left(\overline{\psi}_\mu \gamma^a \psi^b - \overline{\psi}_\mu \gamma^b \psi^a + \overline{\psi}^a \gamma_\mu \psi^b\right)$$

where $\omega_\mu{}^{ab}(e)$ is the same expression we had in the non-supersymmetric case, and indices are changed from flat to curved by using the vierbein e_μ^a and its inverse e^μ_a. The supersymmetry transformations are given by

$$\delta e^a_\mu = \overline{\epsilon}\gamma^a\psi_\mu,$$
$$\delta\omega_\mu^{\ ab} = 0,$$
$$\delta\psi_\mu = \left(\partial_\mu + \frac{1}{4}\omega_\mu^{\ ab}\gamma_{ab}\right)\epsilon.$$

The torsion constraint is not preserved under these transformations. We can preserve the torsion constraint by modifying the $\delta\omega_\mu^{\ ab}$ accordingly. The result is

$$\delta'\omega_{\mu ab} = -\frac{1}{2}e^\nu_a e^\rho_b\left(\overline{\epsilon}\gamma_\rho\psi_{\mu\nu} - \overline{\epsilon}\gamma_\mu\psi_{\nu\rho} + \overline{\epsilon}\gamma_\nu\psi_{\rho\mu}\right).$$

The Lagrangian invariant under the supersymmetry transformations is given by

$$e^{-1}L_{SG} = -\frac{1}{4}R + \overline{\psi}_\mu\gamma^{\mu\nu\rho}\left(\partial_\nu + \frac{1}{4}\omega_\nu^{\ ab}\gamma_{ab}\right)\psi_\rho$$

The proof of invariance is simplified by using that $eR = \frac{1}{2}\epsilon^{\mu\nu\rho\sigma}\epsilon_{abcd}e^a_\mu e^b_\nu R_{\rho\sigma}^{\ \ cd}$ and by writing the varriation with respect to $\omega_\mu^{\ ab}$ in the form $\frac{\delta L}{\delta\omega_\mu^{\ ab}}\delta'\omega_\mu^{\ ab}$. Since $\omega_\mu^{\ ab}$ appears linearly and quadratically one can easily see that, after integrating by parts, its varriation $\frac{\delta L}{\delta\omega_\mu^{\ ab}}$ vanishes if the torsion vanishes and therefore there is no need to substitute the very complicated expression for $\delta'\omega_\mu^{\ ab}$. This method is known as the 1.5 formalism and was used widely in the early formulations of supergravity because it drastically simplifies the proof of the supersymmetry invariance of the proposed actions.

6 Higher dimensional theories

By matching the bosonic and fermionic degrees of freedom we shall determine what are the possible supersymmetric multiplets in all possible dimensions. First a massless vector field in D dimensions have $D-2$ degrees of freedom as the A_0 component does not propagate because it has no time derivatives in the action and one of the transverse components of A_i can be gauged away. For fermions a Dirac spinor has $2^{\left[\frac{D}{2}\right]}$ components, but in certain cases could be subjected to either the Weyl chirality condition or to the Majorana condition or both. The first step is to define the Clifford algebra in dimensions D. The Dirac gammas matrices Γ^M where $M = 0, 1, \cdots, D-1$ will be $2^{\left[\frac{D}{2}\right]} \times 2^{\left[\frac{D}{2}\right]}$ matricess. In even dimensions we can define the gamma matrix $\Gamma_{D+1} = \eta\Gamma^0\Gamma^1\cdots\Gamma^{D-1}$ where $\left(\Gamma_{D+1}\right)^2 = 1$ so that $\eta^2 = (-1)^{s-t} = 1\ (mod 8)$, where s is the number of space-like coordinates and t the number of time like coordinates. For our considerations we will always take $s = D-1$ and $t = 1$. Therefore $\left(\Gamma_{D+1}\right)^2 = 1$ and one can always impose the Weyl condition $\Gamma_{D+1}\psi_\pm = \pm\psi_\pm$. No Weyl fermions can exist in odd dimensions. The Majorana condition $\psi = C\overline{\psi}$ requires the existence of imaginary representations of the gamma matrices in D dimensions so that $\Gamma^{M^*} = -C\Gamma^M C^{-1}$. This is only possible in $D = 2, 3, 4\ (mod 8)$. It is also possible to impose both the Weyl condition and the Majorana condition simultaneously only in $D = 2\ (mod 8)$. Therefore a spinor λ have $2^2 \times \frac{1}{2} = 2$ components in 4 dimensions as one imposes the Majorana condition. (Supersymmetric multiplets are always defined with respect to Majorana spinors so the Weyl condition can only be imposed in dimensions 2 $(mod 8)$. The number of components for λ in 5 dimensions is 4 while in 6 dimensions it is $2^3 \times \frac{1}{2} = 4$ components. In dimensions 7 and 8 it is 8 components. In dimension 9 it is 16 (although in this case it is possible to modify the Majorana condition to reduce it to 8 components). In dimension 10 one can impose simultaneously the Weyl and Majorana conditions so that the number of independent components of λ is 8. We notice that the number of independent components of A_M and λ match in $D = 3, 4, 6$ and 10 and this explains why the super Yang-Mills action is invariant under supersymmetry transformations only in those dimensions. In dimensions higher than 10 the number of independent fermionic components increase much faster than the number of independent components for the vectors A_M and supersymmetric actions for the pure vector multiplet do not exist. This is to be expected because when these higher dimensional theories are analyzed as viewd in four-dimensions, they will correspond to vector multiplets with supersymmetry higher than $N = 4$ which is not possible from our analysis of the supersymmetry representations.

To analyse supergravity theories in higher dimensions we must first determine the number of independent degrees of freedom for the graviton. This is essentially a symmetric metric with $\frac{D(D+1)}{2}$ components where the components g_{0i} do not propagate because they do not acquire second time derivatives. From the other components g_{ij} the D diffeomorphism parameters could be used to reduce the components to $\frac{D(D-3)}{2} = \frac{(D-2)(D-1)}{2} - 1$ so essentially the physical degrees are those of a symmetric traceless metric in $(D-2)$ dimensions. From the Rarita-Schwinger equation for a free field, $\Gamma^{MNP}\partial_N\psi_P = 0$, we can easily show by first contracting this equation with Γ^M and making the gauge choice $\Gamma^M\psi_M = 0$ that $\Gamma^N\partial_N\psi_M = 0$ and $\partial^M\psi_M = 0$. Therefore ψ_M behaves like a massless vector with $D-2$ spinor components and the condition $\Gamma^N\partial_N\psi_M = 0$ eliminates one more spinor component to give finally $(D-3)\,2^{\left[\frac{D}{2}\right]} \times r$ where the reduction factor r is $\frac{1}{2}$ if the Majorana condition is imposed, r is $\frac{1}{4}$ if the Majorana-Weyl condition is imposed and $r = 1$ if no conditions are imposed. From this analysis it is easy to see that the supergravity multiplet in $D = 4$ is the metric (or vierbein) e_μ^a and the gravitino ψ_μ as we have seen before. In five dimensions, the metric has 5 degrees of freedom while the gravitino has 8. A vector field has 3 degrees of freedom so the number of fermionic and bosonic degrees match if we take the multiplet $\left(e_\mu^a, A_\mu, \psi_\mu\right)$. We can classify all possible theories in various dimensions, but it is easier to start with maximal supergravity which is $N = 8$.

6.1 N=1 eleven dimensional supergravity

A spinor in 11 dimensions have $2^{\left[\frac{11}{2}\right]} \times \frac{1}{2} = 16$ degrees of freedom because the Majorana condition could be imposed. As viewed from 4 dimensions where a spinor have 2 degrees of freedom, this corresponds to a spinor with $SO(8)$ symmetry, and according to the classification discussed before, this corresponds to a theory with $N = 8$ supersymmetry in four-dimensions. Therefore $D = 11$ is the highest dimensions where a supergravity theory could be constructed. Also in higher dimensions the number of fermionic degrees increase much faster than those of the metric and other bosonic fields. To determine the field content of the supergravity multiplet in $D = 1$ we note that the metric has 44 degrees of freedom while the gravitino has 128. The difference is 84 degrees which must be present in other bosonic configuration. An antisymmetric tensor of rank p (a p-form) has $\begin{pmatrix} D-2 \\ p \end{pmatrix}$ degrees of freedon, which is equal to 84 when $p = 3$. We deduce that in the multiplet $\left(e_M^A, A_{MNP}, \psi_M\right)$ the bosonic and fermionic degrees of freedom match. This is a necessary but not sufficient condition to have a supersymmetric multiplet and construct a supersymmetric action. This is indeed possible and the full supersymmetric Lagrangian is

$$\begin{aligned} e^{-1}L = &-\frac{1}{4}R - \frac{i}{2}\overline{\psi}_M \Gamma^{MNP} D_N \left(\frac{\omega + \widehat{\omega}}{2}\right)\psi_P - \frac{1}{48}F_{MNPQ}F^{MNPQ} \\ &+ \frac{2e^{-1}}{12^4}\epsilon^{M_1\cdots M_8 PQR} F_{M_1\cdots M_4} F_{M_5\cdots M_8} A_{PQR} \\ &+ \frac{1}{192}\left(\overline{\psi}_M \Gamma^{MNPQRS}\psi_N + 12\overline{\psi}^P \Gamma^{QR}\psi^S\right)\left(F_{PQRS} + \widehat{F}_{PQRS}\right), \end{aligned}$$

where

$$\begin{aligned} F_{MNPQ} &= \frac{1}{4}\left(\partial_M A_{NPQ} - \partial_N A_{PQM} + \partial_P A_{QNM} - \partial_Q A_{MNP}\right), \\ \widehat{\omega}_M^{AB} &= \omega_M^{AB}(e) + \frac{1}{4}\left(\overline{\psi}_M \Gamma^A \psi^B - \overline{\psi}_M \Gamma^B \psi^A + \overline{\psi}^A \Gamma_M \psi^B\right), \\ \widehat{F}_{PQRS} &= F_{PQRS} - 3\overline{\psi}_{[P}\Gamma_{QR}\psi_{S]}. \end{aligned}$$

The action is invariant under the supersymmetry transformations

$$\begin{aligned} \delta e_M^A &= -i\overline{\epsilon}\Gamma^A \psi_M, \\ \delta\psi_M &= \left(\partial_M + \frac{1}{4}\widehat{\omega}_M^{AB}\Gamma_{AB} + \frac{i}{144}\left(\Gamma_M^{\ PQRS} - 8\delta_M^P \Gamma^{QRS}\right)\widehat{F}_{PQRS}\right)\epsilon, \\ \delta A_{MNP} &= \frac{3}{2}\overline{\epsilon}\Gamma_{[MN}\psi_{P]}. \end{aligned}$$

A theory in higher dimensions can produce more complicated content in lower dimensions either by compactification or by dimensional reduction. In dimensional reduction one assumes that the fields are independent of the coordinates in a certain number of internal directions. The action then reduces to that of a lower dimensional theory with more fields. As an exmaple a metric in D dimensions reduces in d dimensions to a metric, $(D-d)$ vectors and $\frac{(D-d)(D-d+1)}{2}$

scalars. A vector in D dimensions reduces in d dimensions to a vector and $(D-d)$ scalars.

6.2 N=1 ten-dimensional Supergravity and Super Yang-Mills

The super Yang-Mills action can be reduced from 10 dimensions to 4. The vector fields A_M^a reduce in 4 dimensions to the vector A_μ^a and the 6 scalars A_i^a, $B_i^a = A_{i+3}^a$, $i = 4, 5, 6$. The spinors $\lambda_{\hat{\alpha}}^a$ in 10 dimensions reduce to four spinors in 4 dimensions $\lambda_{\alpha K}$, $K = 1, \cdots, 4$. The reduced action takes the form

$$I = \int d^4x Tr \Big(-\frac{1}{4} F_{\mu\nu}^a F^{\mu\nu a} + \frac{1}{2} D_\mu A_i^a D^\mu A_i^a + \frac{1}{2} D_\mu B_i^a D^\mu B_i^a + \frac{g}{2} \overline{\lambda}_K \left[\alpha_{KL}^i A_i + i\gamma_5 \beta_{KL}^i B_i, \lambda_L \right] + \frac{g^2}{4} \left([A_i, A_j]^2 + [B_i, B_j]^2 + 2 [A_i, B_j]^2 \right) \Big),$$

where α_{KL}^i and β_{KL}^i are 6 real antisymmetric 4×4 matrices obeying $SU(2) \times SU(2)$ algebra.

Of particular importance is $N = 1$ supergravity in 10 dimensions. The supermultiplet consists of the vielbein e_M^A, an antisymmetric field B_{MN}, a dilaton field ϕ, a gravitino ψ_M and a spinor λ. The supergravity action in ten dimensions can be obtained from the eleven dimensional action by dimensional reduction and consistent truncation. The eleven dimensional vierbein gives in ten dimensions a vielbein, a vector and a scalar dilaton field. The antisymmetric field A_{MNP} gives in ten dimensions antisymmetric tensors of ranks three and two. The gravitino in eleven dimensions give in ten dimensions two gravitinos and two spinors. A consistent truncation is to set the vector, the rank three antisymmetric tensor, one of the gravitinos and one of the spinors to zero. One can show that one of the two supersymmetries will be preserved by this truncation leaving a theory with $N = 1$ supersymmetry. The untruncated theory has $N = 2$ supersymmetry in ten dimensions. The $N = 1$ supergravity Lagrangian is given by

$$\begin{aligned} e^{-1}L = &-\frac{1}{4}R + \frac{1}{12}e^{-2\phi} H_{MNP} H^{MNP} + \frac{1}{2}\partial_M \phi \partial^M \phi \\ &- \frac{i}{2}\overline{\psi}_M \Gamma^{MNP} \left(D_N + \widehat{D}_N \right) \psi_P + \frac{i}{2}\overline{\chi}\Gamma^M \left(D_N + \widehat{D}_N \right) \chi \\ &+ \frac{1}{2\sqrt{2}} \overline{\psi}_M \Gamma^N \Gamma^M \chi \left(\partial_N \phi + \widehat{D}_N \phi \right) \\ &+ \frac{i}{48} \left(\overline{\psi}_M \Gamma^{MNPQR} \psi_R + 6\overline{\psi}^N \Gamma^P \psi^Q - i\sqrt{2}\overline{\psi}_M \Gamma^{NPQ} \Gamma^M \chi \right) \left(H_{NPQ} + \widehat{H}_{NPQ} \right), \end{aligned}$$

where $H_{MNP} = 3\partial_{[M} B_{NP]}$. This action is invariant under the supersymmetry transformations

$$\begin{aligned}
\delta e_M^A &= -i\bar{\epsilon}\Gamma^A\psi_M, \\
\delta\phi &= -\frac{1}{\sqrt{2}}\bar{\epsilon}\chi, \\
\delta B_{MN} &= e^{\phi}\left(i\epsilon\Gamma_{[M}\psi_{N]}\right), \\
\delta\psi_M &= \widehat{D}_M\epsilon - \frac{\sqrt{2}}{48}e^{-\phi}\left(\Gamma_M^{\ NPQ} - 9\delta_M^N\Gamma^{PQ}\right)\widehat{H}_{NPQ}, \\
\delta\chi &= \frac{i}{\sqrt{2}}\Gamma^M\epsilon\widehat{D}_M\phi + \frac{i}{12\sqrt{2}}e^{-\phi}\Gamma^{NPQ}\widehat{F}_{NPQ}.
\end{aligned}$$

The main advantage of the $N = 1$ supergravity Lagrangian in ten dimensions is that it can be coupled to the ten-dimensional super Yang-Mills theory with an arbitrary gauge group. The main modification to the pure supergravity Lagrangian is that the field H_{MNP} is replaced with

$$\begin{aligned}
\widehat{H}_{MNP} &= H_{MNp} - \omega_{MNP}, \\
\omega_{MNP} &= Tr\left(A_{[M}\partial_N A_{P]} + \frac{g}{3}A_{[M}A_N A_{P]}\right),
\end{aligned}$$

where ω_{MNP} is the Chern-Simons form over the gauge Lie algebra. When higher derivative terms are allowed in the supergravity Lagrangian then the modifications to H_{MNP} will also include the Chern-Simons gravitational term. The field $\widehat{H}_{MNP}$ satisfies the equation

$$d\widehat{H} = tr\left(-F\wedge F + R\wedge R\right),$$

where $\widehat{H}$ is a three-form, F is the gauge field strength two-form and R is the curvature tensor two-form.

7 Further directions

The topics considered so far in these lectures would enable us to study some very interesting questions in physics using results from mathematics. For lack of space, these directions will only be described briefly. The interested reader can pursue these problems in the literature.

7.1 Index of Dirac operators

Consider Dirac equation for a massless spinor in ten-dimensions

$$\begin{aligned}
iD_{10}\psi = 0 &= i\Gamma^M D_M\psi = 0, \\
&= i\left(D_4 + D_K\right)\psi,
\end{aligned}$$

where K is the internal 6 dimensional manifold. Let $\Gamma^{(4)} = i\Gamma_0\Gamma_1\Gamma_2\Gamma_3$ and $\Gamma^{(K)} = -i\Gamma_4\Gamma_5\cdots\Gamma_9$ and $\Gamma^{(10)} = \Gamma_0\Gamma_1\cdots\Gamma_9$. This implies that $\left(\Gamma^{(4)}\right)^2 =$

$\left(\Gamma^{(K)}\right)^2 = \left(\Gamma^{(10)}\right)^2 = 1$ and $\Gamma^{(10)} = \Gamma^{(4)}\Gamma^{(K)}$. The Weyl condition on fermions in ten-dimensions $\Gamma^{(10)}\psi = \psi$ is equivalent to $\Gamma^{(4)}\psi = \Gamma^{(K)}\psi$. Defining $\widehat{D}_4 = \Gamma^{(4)}D_4$ and $\widehat{D}_K = \Gamma^{(4)}D_K$ it is easy to see that one can diagonalize simultaneously $\widehat{D}_4$ and $\widehat{D}_K$. Let $H = (iD_K)^2$ and let ψ be an eigenstate with energy E, $H\psi = E\psi$. As H commutes with $\Gamma^{(K)}$ we have

$$H\left(iD_K\psi\right) = E\left(iD_K\psi\right),$$

so that ψ and $D_K\psi$ are degenerate in energy. On the other hand

$$D_K\left(\Gamma^{(K)}\psi\right) = -\Gamma^{(K)}D_K\psi,$$

so that ψ and $D_K\psi$ have opposite chiralities with respect to $\Gamma^{(K)}$ and are therefore linearly independent, unless $D_K\psi = 0$. For every state with $\Gamma^{(K)} = 1$ there is a state with $\Gamma^{(K)} = -1$ except for zero eigenvalues which do not have to be paired. The index of the Dirac operator is then defined by

$$index\left(iD_K\right) = n_+ - n_-$$

where n_+ and n_- are, respectively, the number of zero eigenvalues with eigenvalues $\Gamma^{(K)} = 1$ and $\Gamma^{(K)} = -1$. In the absence of gauge fields the Dirac operator satisfies the properties

$$\left(D_K\psi\right)^* = D_K\psi^*, \qquad \left(\Gamma^{(K)}D_K\psi\right)^* = -\Gamma^{(K)}D_K\psi^*.$$

This is so because in six-dimensions the gamma matrices are real and $\Gamma^{(K)}$ is purely imaginary , and therefore complex conjugation exchanges n_+ and n_-, so that $n_+ = n_-$ and the index is zero. In presence of gauge fields, and for spinors in some representation Q of the gauge group we have

$$index_Q\left(iD_K\right) = -index_{Q^*}\left(iD_K\right)$$

because complex conjugation exchanges the representation Q with its complex conjugate Q^*, and this implies that the index is zero if the representation Q is real or pseudoreal. If Q is complex then the index is not necessarily zero. In terms of the gauge fields and the spin-connection of the underlying manifold the index of the Dirac operator can be computed to be

$$index_Q\left(iD_K\right) = \frac{1}{48\left(2\pi\right)^3}\int\left(tr_Q\left(F\wedge F\wedge F\right) - \frac{1}{8}tr_Q\left(F\right)\wedge tr\left(R\wedge R\right)\right).$$

One can apply this property to explain physical phenomena. We have seen that in the standard model of particle physics, there are three families of particles. In higher dimensional theories it is assumed that particles are in some large representation of a gauge group which is spontaneously broken and the four dimensional theory is obtained by compactification from the higher dimensional one. From the higher dimensional point of view all these particles are massless as they acquire their small masses (compared with the high energy scale of compactification) at a later stage. The number of massless families is then linked to the index of the Dirac operator on the internal manifold. Therefore we can write

$$index_Q\left(iD_K\right) = \frac{1}{2}\chi\left(K\right)$$

where $\chi\left(K\right)$ is the Euler number of the internal manifold. To build a realistic model with three families one must choose the Euler number of the internal manifold to be 6.

7.2 Holonomy and N=1 supersymmetry

At low-energies it is desirable to have a field theory with $N = 1$ supersymmetry in four-dimensions. To have some unbroken supersymmetry, let Q be the supersymmetry generator annihilating the vacuum $|0\rangle$. This implies that

$$\langle 0|\left\{Q, A\right\}|0\rangle = 0,$$

for all operators A. If A is bosonic then $\{Q, A\}$ is fermionic, but $\langle 0|$ fermionic $|0\rangle = 0$ as a fermionic state has no expectation value. If A is fermionic then $\{Q, A\} = \delta A$ and the above relation implies that $\delta A = 0$. In supergravity theories we have seen that the supersymmetry transformations of the gravitino and the other fermions are proportional to the other bosonic fields and the supersymmetry parmeter. For example, assuming that bosonic fields, except for the metric, are set to zero, the gravitino transformation is given by

$$\delta\psi_M = D_M\epsilon = 0.$$

But this is a Killing spinor equation and ϵ is a covariantly constant spinor. This can be satisfied provided Killing spinors exist on the manifold. This also imply consistency conditions

$$\left[D_M, D_N\right]\epsilon = 0,$$

or equivalently $R_{MNPQ}\Gamma^{PQ}\epsilon = 0$. Multiplying by Γ^N and using the property

$$\Gamma^N\Gamma^{PQ} = \Gamma^{NPQ} + g^{NP}\Gamma^Q - g^{NQ}\Gamma^P,$$

where Γ^{NPQ} is completely antisymmetric in the three indices, and using the Riemann identity $R_{M[NPQ]} = o$, we deduce that

$$R_{MQ}\Gamma^Q\epsilon = 0.$$

Let us impose the constraint that the vacuum state of the ten-dimensional space is of the form $M^4 \times K$ where M^4 is a maximally symmetric four-dimensional space and K is a compact six-dimensional manifold. In this case the consistency condition on the Riemann tensor does not admit deSitter or anti deSitter spaces as solutions because in this case

$$R_{\mu\nu\rho\sigma} = \frac{r}{12}\left(g_{\mu\rho}g\nu\sigma - g_{\mu\sigma}g_{\nu\rho}\right)$$

which togother with the consistency condition implies that $r = 0$.

Upon parallel transport around a contractible closed curve γ, a field f is transformed into Uf where

$$U = P \exp_\gamma \int \omega dx$$

where ω is the spin-connection of the n-dimensional internal manifold K. The $SO(n)$ matrices U obtained this way always form a group H called the holonomy group. We can then ask : Under what conditions the manifold K admits a spinor field ϵ obeying $D_i\epsilon = 0$? A covariantly constant spinor ϵ keeps its original value under parallel transport: $U\epsilon = \epsilon$. As an example, let us consider compactifying ten-dimensional supergrravity to four dimensions. In this case the internal manifold K is 6 dimensional. The subgroup of the $SO(6)$ is isomorphic to $SU(4)$ and the left-handed spinor obeying the above condition will be in the fundamental 4-representation. By an $SU(4)$ rotation we can always transform ϵ to the form

$$\epsilon = \begin{pmatrix} 0 \\ 0 \\ 0 \\ \epsilon_0 \end{pmatrix}$$

and this is left invariant by the $SU(3)$ subgroup. We can write the following decomposition

$$\begin{aligned} SO(1,9) &\rightarrow SO(1,3) \times SO(6) \sim SO(1,3) \times SU(4), \\ 16 &= (2,4) \oplus (2',\overline{4}) \end{aligned}$$

where we have decomposed a 16 dimensional spinor of $SO(1,9)$ (the Lorentz group in ten dimensions) in terms of the four-dimensional spinors of $SO(1,3)$ and the 4 and $\overline{4}$ spinor representations of $SO(6)$. From these considerations we conclude that by requiring the manifold K to have $SU(3)$ holonomy, the compactified theory will have $N = 1$ supersymmetry in four-dimensions. Therefore the question to ask now is: What sort of manifold K admits a metric with $SU(3)$ holonomy?

If ϵ is a covariantly constant spinor, then so would be the Kählerian form $k_{ij} = \overline{\epsilon}\Gamma_{ij}\epsilon$, the complex structure $J^i_j = g^{il}k_{lj}$ and the volume form $\omega_{ijk} = \epsilon^T\Gamma_{ijk}\epsilon$. In six-dimensions, if the holonomy group is not $SO(6)$ but $U(3)$ then at any point in K tangent vectors which transform as vectors of $SO(6)$ decompose as $3 + \overline{3}$ under $U(3)$. There is a unique matrix J^i_j acting on tangent vectors in the 3 and $\overline{3}$ representations. J defines an almost complex structure and is invariant under the holonomy group. Therefore it is covariantly constant and its derivatives vanish. A manifold of $U(N)$ holonomy is complex and is also Kähler. If we ask whether it is possible to find a Kähler manifold with $SU(N)$ holonomy instead of $U(N)$ holonomy, then the answer lies in the Calabi-Yau conjecture. This states that a Kähler manifold of vanishing first Chern class admits a Kähler metric of $SU(N)$ holonomy and this metric is unique.

7.3 Solutions of Einstein equations

There are advantages to deal with a supersymmetric system when solving Einstein equations resulting from a coupled gravitational system. This is done by embedding the bosonic system in a supergravity theory, whenever this is possible. One first starts with an ansatz for the metric dictated by the symmetry of the problem. The supersymmetry transformations depend on first order derivatives of the fields. Setting the supersymmetric transformations to zero and requiring that the solution preserve all or part of the supersymmetry, results in a system of first order differential equations, in contrast to the second order differential equations which one have in the original bosonic system. This method has been very successful in finding new solutions for Einstein-coupled systems. To illustrate the power of this method we consider the follwing model.

Consider the coupling of gravity to a dilaton field ϕ and an $SU(2)$ Yang-Mills field A^a_μ. described by the action

$$S = \int \left(-\frac{1}{4}\, R + \frac{1}{2}\, \partial_\mu \phi\, \partial^\mu \phi - \frac{1}{4}\, e^{2\phi}\, F^a_{\mu\nu} F^{a\mu\nu} + \frac{1}{8}\, e^{-2\phi} \right) \sqrt{-\mathbf{g}}\, d^4x,$$

where $F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + \varepsilon_{abc} A^b_\mu A^c_\nu$, and the dilaton potential can be viewed as an effective negative, position-dependent cosmological term $\Lambda(\phi) = -\frac{1}{4} e^{-2\phi}$.

We shall consider static, spherically symmetric, purely magnetic configurations of the bosonic fields, and for this we parameterize the fields as follows:

$$ds^2 = N\sigma^2 dt^2 - \frac{dr^2}{N} - r^2(d\theta^2 + \sin^2\theta\, d\varphi^2),$$

$$\alpha^a A^a_\mu dx^\mu = w\ (-\alpha^2\, d\theta + \alpha^1\, \sin\theta\, d\varphi) + \alpha^3\, \cos\theta\, d\varphi,$$

where N, σ, w, as well as the dilaton ϕ, are functions of the radial coordinate r. The field equations, following from the action, read

$$(rN)' + r^2 N \phi'^{\,2} + U + r^2\Lambda(\phi) = 1,$$

$$\left(\sigma N r^2 \phi'\right)' = \sigma\, (U - r^2\Lambda(\phi)),$$

$$r^2\left(N\sigma e^{2\phi}\, w'\right)' = \sigma e^{2\phi}\, w(w^2 - 1),$$

$$\sigma' = \sigma\, (r\phi'^{\,2} + 2e^{2\phi}\, w'^{\,2}/r),$$

where $U = 2e^{2\phi}\left(Nw'^{\,2} + (w^2-1)^2/2r^2\right)$. This is a system of second order differential equations which could not be solved. We now adopt the strategy of embedding this action into a supersymmetric action and use proporties of supersymmetry to find a solution.

The action of the N=4 gauged SU(2)×SU(2) supergravity includes a vierbein e^α_μ, four Majorana spin-3/2 fields $\psi_\mu \equiv \psi^{\rm I}_\mu$ $({\rm I} = 1, \ldots 4)$, vector and pseudovector non-Abelian gauge fields $A^{(1)\,a}_\mu$ and $A^{(2)\,a}_\mu$ with independent gauge coupling constants g_1 and g_2, respectively, four Majorana spin-1/2 fields $\chi \equiv \chi^{\mathrm{I}}$, the axion $\mathbf{a}$ and the dilaton ϕ . The bosonic part of the action reads

$$S = \int \Bigg(-\frac{1}{4}\, R + \frac{1}{2}\, \partial_\mu \phi\, \partial^\mu \phi + \frac{1}{2}\, e^{-4\phi} \partial_\mu \mathbf{a}\, \partial^\mu \mathbf{a} - \frac{1}{4}\, e^{2\phi} \sum_{s=1}^{2} F^{(s)\,a}_{\mu\nu} F^{(s)\,a\mu\nu}$$

$$-\frac{1}{2}\, \mathbf{a} \sum_{s=1}^{2} F^{(s)\,a}_{\mu\nu} * F^{(s)\,a\mu\nu} + \frac{g^2}{8}\, e^{-2\phi} \Bigg) \sqrt{-\mathbf{g}}\, d^4x.$$

Here $g^2 = g_1^2 + g_2^2$, the gauge field tensor $F^{(s)\,a}_{\mu\nu} = \partial_\mu A^{(s)\,a}_\nu - \partial_\nu A^{(s)\,a}_\mu + g_C\, \varepsilon_{abc}\, A^{(s)\,b}_\mu A^{(s)\,c}_\nu$ (there is no summation over $s = 1, 2$), and $*F^{(s)\,a}_{\mu\nu}$ is the dual tensor. The dilaton potential can be viewed as an effective negative, position-dependent cosmological term $\Lambda(\phi) = -\frac{1}{8}\, g^2\, e^{-2\phi}$. The ungauged version of the theory corresponds to the case where $g_1 = g_2 = 0$. We consider the truncated theory specified by the conditions $g_2 = A^{a(2)}_\nu = 0$. In addition, we require the vector field A^a_μ to be purely magnetic, which allows us to set the axion to zero.

For a purely bosonic configuration, the supersymmetry transformation laws are

$$\delta\bar{\chi} = -\frac{i}{\sqrt{2}}\, \bar{\epsilon}\, \gamma^\mu \partial_\mu \phi - \frac{1}{2} e^{\phi}\, \bar{\epsilon}\, \alpha^a F^a_{\mu\nu}\, \sigma^{\mu\nu} + \frac{1}{4}\, e^{-\phi}\, \bar{\epsilon},$$

$$\delta\bar{\psi}_\rho = \bar{\epsilon} \left(\overleftarrow{\partial}_\rho - \frac{1}{2}\, \omega_{\rho mn}\, \sigma^{mn} + \frac{1}{2}\, \alpha^a A^a_\rho \right) - \frac{1}{2\sqrt{2}}\, e^{\phi}\, \bar{\epsilon}\, \alpha^a F^a_{\mu\nu}\, \gamma_\rho\, \sigma^{\mu\nu} + \frac{i}{4\sqrt{2}}\, e^{-\phi}\, \bar{\epsilon}\, \gamma_\rho,$$

the variations of the bosonic fields being zero. In these formulas, $\epsilon \equiv \epsilon^{\mathrm{I}}$ are four Majorana spinor supersymmetry parameters, $\alpha^a \equiv \alpha^a_{\mathrm{IJ}}$ are the SU(2) gauge group generators, and $\omega_{\rho mn}$ is the tetrad connection.

The field configuration is supersymmetric, provided that there are non-trivial supersymmetry Killing spinors ϵ for which the variations of the fermion fields vanish. Specifying to the above configuration and putting $\delta\bar{\chi} = \delta\bar{\psi}_\mu = 0$, the supersymmetry constraints become a system of equations for the four spinors ϵ^{I}. The procedure which solves these equations is rather involved. The consistency of the algebraic constraints requires that the determinants of the corresponding coefficient matrices vanish and that the matrices commute with each other. These consistency conditions can be expressed by the following relations for the background:

$$N\sigma^2 = e^{2(\phi - \phi_0)},$$

$$N = \frac{1 + w^2}{2} + e^{2\phi}\, \frac{(w^2 - 1)^2}{2r^2} + \frac{r^2}{8} e^{-2\phi},$$

$$r\phi' = \frac{r^2}{8N} e^{-2\phi} \left(1 - 4e^{4\phi}\, \frac{(w^2 - 1)^2}{r^4} \right),$$

$$rw' = -2w \frac{r^2}{8N} e^{-2\phi} \left(1 + 2e^{2\phi} \frac{w^2 - 1}{r^2} \right),$$

with constant ϕ_0. Under these conditions, the solution of the algebraic constraints yields ϵ in terms of only two independent functions of r. The remaining differential constraint then uniquely specify these two functions up to two integration constants, which finally corresponds to two unbroken supersymmetries. Introducing the new variables $x = w^2$ and $R^2 = \frac{1}{2}r^2 e^{-2\phi}$, the above equations become equivalent to one first order differential equation

$$2xR\,(R^2 + x - 1)\frac{dR}{dx} + (x+1)\,R^2 + (x-1)^2 = 0.$$

If $R(x)$ is known, the radial dependence of the functions, $x(r)$ and $R(r)$, can be determined. Define the following substitution:

$$x = \rho^2\, e^{\xi(\rho)}, \qquad R^2 = -\rho\frac{d\xi(\rho)}{d\rho} - \rho^2\, e^{\xi(\rho)} - 1,$$

where $\xi(\rho)$ is obtained from

$$\frac{d^2\xi(\rho)}{d\rho^2} = 2\, e^{\xi(\rho)}.$$

The most general (up to reparametrizations) solution of this equation which ensures that $R^2 > 0$ is $\xi(\rho) = -2\ln\sinh(\rho - \rho_0)$. This gives us the general solution. The metric is non-singular at the origin if only $\rho_0 = 0$, in which case

$$R^2(\rho) = 2\rho\coth\rho - \frac{\rho^2}{\sinh^2\rho} - 1,$$

one has $R^2(\rho) = \rho^2 + O(\rho^4)$ as $\rho \to 0$, and $R^2(\rho) = 2\rho + O(1)$ as $\rho \to \infty$. The last step is to obtain $r(s)$, which finally gives us a family of completely regular solutions of the Bogomolny equations:

$$ds^2 = a^2\,\frac{\sinh\rho}{R(\rho)}\left\{dt^2 - d\rho^2 - R^2(\rho)(d\vartheta^2 + \sin^2\vartheta d\varphi^2)\right\},$$

$$w = \pm\frac{\rho}{\sinh\rho}, \qquad e^{2\phi} = a^2\,\frac{\sinh\rho}{2\,R(\rho)},$$

where $0 \le \rho < \infty$, and we have chosen $2\phi_0 = -\ln 2$. The geometry described by the above line element is everywhere regular, the coordinates covering the whole space whose topology is R^4. The geometry becomes flat at the origin, but asymptotically it is not flat, even though the cosmological term $\Lambda(\phi)$ vanishes at infinity. In the asymptotic region all curvature invariants tend to zero, however, not fast enough. The Schwarzschild metric functions for $r \to \infty$ are $N \propto \ln r$ and $N\sigma^2 \propto r^2/4\ln r$, the non-vanishing Weyl tensor invariant being $\Psi_2 \propto -1/6r^2$.

References

[1] S. Weinberg, *Quantum Field Theory*, volume 1, Cambridge University Press, 1998.