Contents

1 Introduction

Symmetries play an important role in particle physics. Continuous (and local) symmetries such as Lorentz, Poincare and gauge symmetries are essential to understand several phenomena, which happen in particle physics like strong, weak and electromagnetic interactions among particles. Discrete symmetries such as C, P and T are also important.

Furthermore, Abelian discrete symmetries, Z_N, are also often imposed in order to control allowed couplings in model building for particle physics, in particular model building beyond the standard model. For example, R-parity and matter parities are assumed in supersymmetric standard models to forbid the fast proton decay. Such parities are also important from the viewpoint of dark matter. In addition to Abelian discrete symmetries, non-Abelian discrete symmetries were applied for model building of particle physics recently, in particular to understand the flavor physics.

There are many free parameters in the standard model including its extension with neutrino mass terms and most of them are originated from the flavor sector, i.e. Yukawa couplings of quarks and leptons. The flavor symmetries are introduced to control Yukawa couplings in the three generations although the origin of the generations is unknown. The quark masses and mixing angles have been discussed in the standpoint of the flavor symmetries. The discovery of neutrino masses and the neutrino mixing [1, 2] has stimulated the work of the flavor symmetries. Recent experiments of the neutrino oscillation go into a new phase of precise determination of mixing angles and mass squared differences [3, 4, 5, 6], which indicate the tri-bimaximal mixing for three flavors in the lepton sector [7, 8, 9, 10]. These large mixing angles are completely different from the quark mixing ones. Therefore, it is very important to find a natural model that leads to these mixing patterns of quarks and leptons with good accuracy. Non-Abelian discrete symmetries are studied to apply for flavor physics, that is, model building to derive experimental values of quark/lepton masses and mixing angles by assuming non-Abelian discrete flavor symmetries of quarks and leptons. Especially, the lepton mixing has been intensively discussed in non-Abelian discrete flavor symmetries as seen, e.g. in the review by Altarelli and Feruglio [11].

The flavor symmetry may be a remnant of the higher dimensional space-time symmetry, after it is broken down to the 4-dimensional Poincare symmetry through compactification, e.g. via orbifolding. Actually, it was shown how the flavor symmetry A_4 (or S_4) can arise if the three fermion generations are taken to live on the fixed points of a specific 2-dimensional orbifold [12]. Further non-Abelian discrete symmetries can arise in a similar setup [13].

Superstring theory is a promising candidate for unified theory including gravity. Certain string modes correspond to gauge bosons, quarks, leptons, Higgs bosons and gravitons as well as their superpartners. Superstring theory predicts six extra dimensions. Certain classes of discrete symmetries can be derived from superstring theories. A combination among geometrical symmetries of a compact space and stringy selection rules for couplings enhances discrete flavor symmetries. For example, D_4 and $\Delta(54)$ flavor symmetries can be obtained in heterotic orbifold models [14, 15, 16]. In addition to these flavor symmetries, the $\Delta(27)$ flavor symmetry can be derived from magnetized/intersecting D-brane models [17, 18, 19].

There is another possibility that non-Abelian discrete groups are originated from the breaking of continuous (gauge) flavor symmetries [20, 21, 22].

Thus, the non-Abelian discrete symmetry can arise from the underlying theory, e.g. the string theory or compactification via orbifolding. Also, the non-Abelian discrete-symmetries are interesting tools for controlling the flavor structure in model building from the bottom-up approach. Hence, the non-Abelian flavor symmetries could become a bridge between the low-energy physics and the underlying theory. Therefore, it is quite important to study the properties of non-Abelian groups.

Non-Abelian continuous groups are well-known and of course there are several good reviews and books. On the other hand, non-Abelian discrete symmetries may not be familiar to all of particle physicists compared with non-Abelian continuous symmetries. However, non-Abelian discrete symmetries have become important tools for model building, in particular for the flavor physics. Our purpose of this article is to review pedagogically non-Abelian discrete groups with minding particle phenomenology and show group-theoretical aspects for many concrete groups explicitly, such as representations and their tensor products [23]-[28]. We show these aspects in detail for S_N [29]-[88], A_N [89]-[155], T' [27],[156]-[164], D_N [165]-[179], Q_N [180]-[185], $\Sigma(2N^2)$ [186], $\Delta(3N^2)$ [187]-[194], T_7[195], $\Sigma(3N^3)$ [195], and $\Delta(6N^2)$ groups [194],[196]-[198]. We explain pedagogically how to derive conjugacy classes, characters, representations and tensor products for these groups (with a finite number) when algebraic relations are given. Thus, the readers could apply for other groups.

In applications for particle physics, the breaking patterns of discrete groups and decompositions of multiplets are also important. Such aspects are studied in this paper.

Symmetries at the tree level are not always symmetries in quantum theory. If symmetries are anomalous, breaking terms are induced by quantum effects. Such anomalies are important in applications for particle physics. Here, we study such anomalies for discrete symmetries [27],[199]-[212] and show anomaly-free conditions explicitly for the above concrete groups. If flavor symmetries are stringy symmetries, these anomalies may also be controlled by string dynamics, i.e. anomaly cancellation.

This article is organized as follows. In section 2, we summarize basic group-theoretical aspects, which are necessary in the rest of sections. The readers, which are familiar to group theory, can skip section 2. In sections 3 to 12, we present non-Abelian discrete groups, S_N, A_N, T', D_N, Q_N, $\Sigma(2N^2)$, $\Delta(3N^2)$, T_7, $\Sigma(3N^3)$, and $\Delta(6N^2)$, respectively. In section 13, the breaking patterns of the non-Abelian discrete groups are discussed. In section 14, we review the anomaly of non-Abelian flavor symmetries, which is a topic in the particle physics, and show the anomaly-free conditions explicitly for the above concrete groups. In section 15, typical flavor models with the non-Abelian discrete symmetries are presented. Section 16 is devoted to summary. In appendix A, useful theorems on finite group theory are presented. In appendices B and C, we show representation bases of S_4 and A_4, respectively, which are different from those in sections 3 and 4.

2 Finite groups

In this section, we summarize basic aspects on group theory, which are necessary in the following sections. We use several theorems without their proofs, in order for the readers to read easily. However, proofs of useful theorems are given in Appendix A. (See also e.g. Refs. [23, 24, 26, 28].)

A group, G, is a set, where multiplication is defined such that the following properties are satisfied:

1. **Closure**

   If a and b are elements of the group G, $c = ab$ is also its element.

2. **Associativity**

   $(ab)c = a(bc)$ for $a, b, c \in G$.

3. **Identity**

   The group G includes an identity element e, which satisfies $ae = ea = a$ for any element $a \in G$.

4. **Inverse**

   The group G includes an inverse element a^{-1} for any element $a \in G$ such that $aa^{-1} = a^{-1}a = e$.

The **order** is the number of elements in G. The order of a finite group is finite. The group G is called **Abelian** if all of their elements are commutable each other, i.e. $ab = ba$. If all of elements do not satisfy the commutativity, the group is called **non-Abelian**. One of simple finite groups is the cyclic group Z_N, which consists of

$$\{e, a, a^2, \cdots, a^{N-1}\}, \tag{1}$$

where $a^N = e$. The Z_N group can be represented as discrete rotations, whose generator a corresponds to $2\pi/N$ rotation. The Z_N group is Abelian. We focus on non-Abelian discrete symmetries in the following sections.

If a subset H of the group G is also a group, H is called the **subgroup** of G. The order of the subgroup H must be a divisor of the order of G. That is **Lagrange's theorem**. (See Appendix A.) If a subgroup N of G satisfies $g^{-1}Ng = N$ for any element $g \in G$, the subgroup N is called a **normal subgroup** or an **invariant subgroup**. The subgroup H and normal subgroup N of G satisfy $HN = NH$ and it is a subgroup of G, where HN denotes

$$\{h_i n_j | h_i \in H, n_j \in N\}, \tag{2}$$

and NH denotes a similar meaning.

When $a^h = e$ for an element $a \in G$, the number h is called the **order** of a. The elements, $\{e, a, a^2, \cdots, a^{h-1}\}$, form a subgroup, which is the Abelian Z_h group with the order h.

The elements $g^{-1}ag$ for $g \in G$ are called elements conjugate to the element a. The set including all elements to conjugate to an element a of G, $\{g^{-1}ag, \ \forall g \in G\}$, is called a **conjugacy class**. All of elements in a conjugacy class have the same order since

$$(gag^{-1})^h = ga(g^{-1}g)a(g^{-1}g) \cdots ag^{-1} = ga^h g^{-1} = geg^{-1} = e. \tag{3}$$

The conjugacy class including the identity e consists of the single element e.

We consider two groups, G and G', and a map f of G on G'. This map is **homomorphic** only if the map preserves the multiplication structure, that is,

$$f(a)f(b) = f(ab), \tag{4}$$

for $a, b \in G$. Furthermore, the map is **isomorphic** when the map is one-to-one correspondence.

A **representation** of G is a homomorphic map of elements of G onto matrices, $D(g)$ for $g \in G$. The representation matrices should satisfy $D(a)D(b) = D(c)$ if $ab = c$ for $a, b, c \in G$. The vector space v_j, on which representation matrices act, is called a **representation space** such as $D(g)_{ij}v_j$ $(j = 1, \cdots, n)$. The dimension n of the vector space v_j $(j = 1, \cdots, n)$ is called as a **dimension** of the representation. A subspace in the representation space is called **invariant subspace** if $D(g)_{ij}v_j$ for any vector v_j in the subspace and any element $g \in G$ also corresponds to a vector in the same subspace. If a representation has an invariant subspace, such a representation is called **reducible**. A representation is **irreducible** if it has no invariant subspace. In particular, a representation is called **completely reducible** if $D(g)$ for $g \in G$ are written as the following block diagonal form,

$$\begin{pmatrix} D_1(g) & 0 & & \\ 0 & D_2(g) & & \\ & & \ddots & \\ & & & D_r(g) \end{pmatrix}, \tag{5}$$

where each $D_\alpha(g)$ for $\alpha = 1, \cdots, r$ is irreducible. This implies that a reducible representation $D(g)$ is the direct sum of $D_\alpha(g)$,

$$\sum_{\alpha=1}^{r} \oplus D_\alpha(g). \tag{6}$$

Every (reducible) representation of a fine group is completely reducible. Furthermore, every representation of a fine group is equivalent to a unitary representation.(See Appendix A.) The simplest (irreducible) representation is found that $D(g) = 1$ for all elements g, that is, a trivial singlet. The matrix representations satisfy the following orthogonality relation,

$$\sum_{g \in G} D_\alpha(g)_{i\ell} D_\beta(g^{-1})_{mj} = \frac{N_G}{d_\alpha} \delta_{\alpha\beta}\delta_{ij}\delta_{\ell m}, \tag{7}$$

where N_G is the order of G and d_α is the dimension of the $D_\alpha(g)$.(See Appendix A.)

The **character** $\chi_D(g)$ of a representation $D(g)$ is the trace of the representation matrix,

$$\chi_D(g) = \text{tr } D(g) = \sum_{i=1}^{d_\alpha} D(g)_{ii}. \tag{8}$$

The element conjugate to a has the same character because of the property of the trace,

$$\text{tr } D(g^{-1}ag) = \text{tr } \left(D(g^{-1})D(a)D(g)\right) = \text{tr } D(a), \tag{9}$$

that is, the characters are constant in a conjugacy class. The characters satisfy the following orthogonality relation,

$$\sum_{g \in G} \chi_{D_\alpha}(g)^* \chi_{D_\beta}(g) = N_G \delta_{\alpha\beta}, \tag{10}$$

where N_G denotes the order of a group G. (See Appendix A.) That is, the characters of different irreducible representations are orthogonal and different from each others. *Furthermore it is found that the number of irreducible representations must be equal to the number of conjugacy classes.* (See Appendix A.) In addition, they satisfy the following orthogonality relation,

$$\sum_{\alpha} \chi_{D_\alpha}(g_i)^* \chi_{D_\alpha}(g_j) = \frac{N_G}{n_i} \delta_{C_i C_j}, \tag{11}$$

where C_i denotes the conjugacy class of g_i and n_i denotes the number of elements in the conjugacy class C_i. (See Appendix A.) That is, the above equation means that the right hand side is equal to $\frac{N_G}{n_i}$ if g_i and g_j belong to the same conjugacy class, and that otherwise it must vanish. A trivial singlet, $D(g) = 1$ for any $g \in G$, must always be included. Thus, the corresponding character satisfies $\chi_1(g) = 1$ for any $g \in G$.

Suppose that there are m_n n-dimensional irreducible representations, that is, $D(g)$ are represented by $(n \times n)$ matrices. The identity e is always represented by the $(n \times n)$ identity matrix. Obviously, the character $\chi_{D_\alpha}(C_1)$ for the conjugacy class $C_1 = \{e\}$ is found that $\chi_{D_\alpha}(C_1) = n$ for the n-dimensional representation. Then, the orthogonality relation (11) requires

$$\sum_{\alpha} [\chi_\alpha(C_1)]^2 = \sum_n m_n n^2 = m_1 + 4m_2 + 9m_3 + \cdots = N_G, \tag{12}$$

where $m_n \geq 0$. Furthermore, m_n must satisfy

$$\sum_n m_n = \text{the number of conjugacy classes}, \tag{13}$$

because the number of irreducible representations is equal to the number of conjugacy classes. Eqs. (12) and (13) as well as Eqs. (10) and (11) are often used in the following sections to determine characters.

We can construct a lager group from more than two groups, G_i, by a certain product. A rather simple one is the **direct product**. We consider e.g. two groups G_1 and G_2. Their direct product is denoted as $G_1 \times G_2$, and its multiplication rule is defined as

$$(a_1, a_2)(b_1, b_2) = (a_1 b_1, a_2 b_2), \tag{14}$$

for $a_1, b_1 \in G_1$ and $a_2, b_2 \in G_2$.

The **semi-direct product** is more non-trivial product between two groups G_1 and G_2, and it is defined such as

$$(a_1, a_2)(b_1, b_2) = (a_1 f_{a_2}(b_1), a_2 b_2), \tag{15}$$

for $a_1, b_1 \in G_1$ and $a_2, b_2 \in G_2$, where $f_{a_2}(b_1)$ denotes a homomorphic map from G_2 to G_1. This semi-direct product is denoted as $G_1 \rtimes_f G_2$. We consider the group G, its subgroup H, and normal subgroup N, whose elements are h_i and n_j, respectively. When $G = NH = HN$ and $N \cap H = \{e\}$, the semi-direct product $N \rtimes_f H$ is isomorphic to G, $G \simeq N \rtimes_f H$, where we use the map f as

$$f_{h_i}(n_j) = h_i n_j (h_i)^{-1}. \tag{16}$$

For the notation of the semi-direct product, we will often omit f and denote it as $N \rtimes H$.

3 S_N

All possible permutations among N objects x_i with $i = 1, \cdots, N$, form a group,

$$(x_1, \cdots, x_n) \to (x_{i_1}, \cdots, x_{i_N}). \tag{17}$$

This group is the so-called S_N with the order $N!$, and S_N is often called as the symmetric group. In the following we show concrete aspects on S_N for smaller N. The simplest one of S_N except the trivial S_1 is S_2, which consists of two permutations,

$$(x_1, x_2) \to (x_1, x_2), \qquad (x_1, x_2) \to (x_2, x_1). \tag{18}$$

This is nothing but Z_2, that is Abelian. Thus, we start with S_3.

3.1 S_3

S_3 consists of all permutations among three objects, (x_1, x_2, x_3) and its order is equal to $3! = 6$. All of six elements correspond to the following transformations,

$$\begin{aligned}
e &: (x_1, x_2, x_3) \to (x_1, x_2, x_3), \\
a_1 &: (x_1, x_2, x_3) \to (x_2, x_1, x_3), \\
a_2 &: (x_1, x_2, x_3) \to (x_3, x_2, x_1), \\
a_3 &: (x_1, x_2, x_3) \to (x_1, x_3, x_2), \\
a_4 &: (x_1, x_2, x_3) \to (x_3, x_1, x_2), \\
a_5 &: (x_1, x_2, x_3) \to (x_2, x_3, x_1).
\end{aligned} \tag{19}$$

Their multiplication forms a closed algebra, e.g.

$$\begin{aligned}
a_1 a_2 &: (x_1, x_2, x_3) \to (x_2, x_3, x_1), \\
a_2 a_1 &: (x_1, x_2, x_3) \to (x_3, x_1, x_2), \\
a_4 a_2 &: (x_1, x_2, x_3) \to (x_1, x_3, x_2),
\end{aligned} \tag{20}$$

i.e.

$$a_1 a_2 = a_5, \quad a_2 a_1 = a_4, \quad a_4 a_2 = a_2 a_1 a_2 = a_3. \tag{21}$$

Thus, by defining $a_1 = a, a_2 = b$, all of elements are written as

$$\{e, a, b, ab, ba, bab\}. \tag{22}$$

Note that $aba = bab$. The S_3 group is a symmetry of an equilateral triangle as shown in Figure 1. The elements a and ab correspond to a reflection and the $2\pi/3$ rotation, respectively.

- **Conjugacy classes**

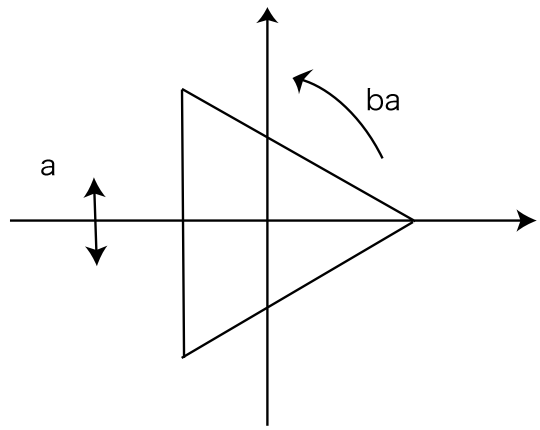

Figure 1: The S_3 symmetry of an equilateral triangle

These elements are classified to three conjugacy classes,

$$C_1 : \{e\}, \quad C_2 : \{ab, ba\}, \quad C_3 : \{a, b, bab\}. \tag{23}$$

Here, the subscript of C_n, n, denotes the number of elements in the conjugacy class C_n. Their orders are found as

$$(ab)^3 = (ba)^3 = e, \qquad a^2 = b^2 = (bab)^2 = e. \tag{24}$$

The elements $\{e, ab, ba\}$ correspond to even permutations, while the elements $\{a, b, bab\}$ are odd permutations.

• Characters and representations

Let us study irreducible representations of S_3. The number of irreducible representations must be equal to three, because there are three conjugacy classes. We assume that there are m_n n-dimensional representations, that is, $D(g)$ are represented by $(n \times n)$ matrices. Here, m_n must satisfy $\sum_n m_n = 3$. Furthermore, the orthogonality relation (12) requires

$$\sum_\alpha [\chi_\alpha(C_1)]^2 = \sum_n m_n n^2 = m_1 + 4m_2 + 9m_3 + \cdots = 6, \tag{25}$$

where $m_n \geq 0$. This equation has only two possible solutions, $(m_1, m_2) = (2, 1)$ and $(6, 0)$, but only the former $(m_1, m_2) = (2, 1)$ satisfies $m_1 + m_2 = 3$. Thus, irreducible representations of S_3 include two singlets $\mathbf{1}$ and $\mathbf{1'}$, and a doublet $\mathbf{2}$. We denote their characters by $\chi_1(g), \chi_{1'}(g)$ and $\chi_2(g)$, respectively. Obviously, it is found that $\chi_1(C_1) = \chi_{1'}(C_1) = 1$ and $\chi_2(C_1) = 2$. Furthermore, one of singlet representations corresponds to a trivial singlet, that is, $\chi_1(C_2) = \chi_1(C_3) = 1$. The characters, which are not fixed at this stage, are $\chi_{1'}(C_2)$, $\chi_{1'}(C_3)$, $\chi_2(C_2)$ and $\chi_2(C_3)$. Now let us determine them. For a non-trivial singlet $\mathbf{1'}$, representation matrices are nothing but characters, $\chi_{1'}(C_2)$ and $\chi_{1'}(C_3)$. They must satisfy

$$(\chi_{1'}(C_2))^3 = 1, \qquad (\chi_{1'}(C_3))^2 = 1. \tag{26}$$

	h	χ_1	$\chi_{1'}$	χ_2
C_1	1	1	1	2
C_2	3	1	1	-1
C_3	2	1	-1	0

Table 1: Characters of S_3 representations

Thus, $\chi_{1'}(C_2)$ is one of 1, ω and ω^2, where $\omega = \exp[2\pi i/3]$, and $\chi_{1'}(C_3)$ is 1 or -1. On top of that, the orthogonality relation (10) requires

$$\sum_g \chi_1(g)\chi_{1'}(g) = 1 + 2\chi_{1'}(C_2) + 3\chi_{1'}(C_3) = 0. \tag{27}$$

Its unique solution is obtained by $\chi_{1'}(C_2) = 1$ and $\chi_{1'}(C_3) = -1$. Furthermore, the orthogonality relations (10) and (11) require

$$\begin{aligned} \sum_g \chi_1(g)\chi_2(g) &= 2 + 2\chi_2(C_2) + 3\chi_2(C_3) = 0, \qquad (28) \\ \sum_\alpha \chi_\alpha(C_1)^*\chi_\alpha(C_2) &= 1 + \chi_{1'}(C_2) + 2\chi_2(C_2) = 0. \qquad (29) \end{aligned}$$

Their solution is written by $\chi_2(C_2) = -1$ and $\chi_2(C_3) = 0$. These results are shown in Table 1.

Next, let us figure out representation matrices $D(g)$ of S_3 by using the character in Table 1. For singlets, their characters are nothing but representation matrices. Thus, let us consider representation matrices $D(g)$ for the doublet, where $D(g)$ are (2×2) unitary matrices. Obviously, $D_2(e)$ is the (2×2) identity matrices. Because of $\chi_2(C_3) = 0$, one can diagonalize one element of the conjugacy class C_3. Here we choose e.g. a in C_3 as the diagonal element,

$$a = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{30}$$

The other elements in C_3 as well as C_2 are non-diagonal matrices. Recalling $b^2 = e$, we can write

$$b = \begin{pmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{pmatrix}, \quad bab = \begin{pmatrix} \cos 2\theta & \sin 2\theta \\ \sin 2\theta & -\cos 2\theta \end{pmatrix}. \tag{31}$$

Then, we can write elements in C_2 as

$$ab = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}, \quad ba = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}. \tag{32}$$

Recall that the trace of elements in C_2 is equal to -1. Then, it is found that $\cos\theta = -1/2$, that is, $\theta = 2\pi/3, 4\pi/3$. When we choose $\theta = 4\pi/3$, we obtain the matrix representation of S_3 as

$$e = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad a = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad b = \begin{pmatrix} -\frac{1}{2} & -\frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix},$$
$$ab = \begin{pmatrix} -\frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix}, \quad ba = \begin{pmatrix} -\frac{1}{2} & \frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix}, \quad bab = \begin{pmatrix} -\frac{1}{2} & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix}. \tag{33}$$

• **Tensor products**

Finally, we consider tensor products of irreducible representations. Let us start with the tensor products of two doublets, (x_1, x_2) and (y_1, y_2). For example, each element $x_i y_j$ is transformed under b as

$$\begin{aligned}
x_1y_1 &\rightarrow \frac{x_1y_1 + 3x_2y_2 + \sqrt{3}(x_1y_2 + x_2y_1)}{4}, \\
x_1y_2 &\rightarrow \frac{\sqrt{3}x_1y_1 - \sqrt{3}x_2y_2 - x_1y_2 + 3x_2y_1}{4}, \\
x_2y_1 &\rightarrow \frac{\sqrt{3}x_1y_1 - \sqrt{3}x_2y_2 - x_2y_1 + 3x_1y_2}{4}, \\
x_2y_2 &\rightarrow \frac{3x_1y_1 + x_2y_2 - \sqrt{3}(x_1y_2 + x_2y_1)}{4}.
\end{aligned} \tag{34}$$

Thus, it is found that

$$b(x_1y_1 + x_2y_2) = (x_1y_1 + x_2y_2), \quad b(x_1y_2 - x_2y_1) = -(x_1y_2 - x_2y_1). \tag{35}$$

That implies these linear combinations correspond to the singlets,

$$\mathbf{1} : x_1y_1 + x_2y_2, \quad \mathbf{1}' : x_1y_2 - x_2y_1. \tag{36}$$

Furthermore, it is found that

$$b\begin{pmatrix} x_2y_2 - x_1y_1 \\ x_1y_2 + x_2y_1 \end{pmatrix} = \begin{pmatrix} -\frac{1}{2} & -\frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix} \begin{pmatrix} x_2y_2 - x_1y_1 \\ x_1y_2 + x_2y_1 \end{pmatrix}. \tag{37}$$

Hence, $(x_2y_2 - x_2y_2, x_1y_2 + x_2y_1)$ corresponds to the doublet, i.e.

$$\mathbf{2} = \begin{pmatrix} x_2y_2 - x_1y_1 \\ x_1y_2 + x_2y_1 \end{pmatrix}. \tag{38}$$

Similarly, we can study the tensor product of the doublet (x_1, x_2) and the singlet $\mathbf{1}'$ y'. Their products $x_i y'$ transform under b as

$$\begin{aligned}
x_1y' &\rightarrow \frac{1}{2}x_1y' + \frac{\sqrt{3}}{2}x_2y', \\
x_2y' &\rightarrow \frac{\sqrt{3}}{2}x_1y' - \frac{1}{2}x_2y'.
\end{aligned} \tag{39}$$

That implies those form a doublet,

$$\mathbf{2}:\begin{pmatrix} -x_2y' \\ x_1y' \end{pmatrix}. \tag{40}$$

These results are summarized as follows,

$$\begin{aligned}
&\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}_{\mathbf{2}} \otimes \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}_{\mathbf{2}} = (x_1y_1+x_2y_2)_{\mathbf{1}} + (x_1y_2-x_2y_1)_{\mathbf{1}'} + \begin{pmatrix} x_1y_2+x_2y_1 \\ x_1y_1-x_2y_2 \end{pmatrix}_{\mathbf{2}}, \\
&\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}_{\mathbf{2}} \otimes (y')_{\mathbf{1}'} = \begin{pmatrix} -x_2y' \\ x_1y' \end{pmatrix}_{\mathbf{2}}, \\
&(x')_{\mathbf{1}'} \otimes (y')_{\mathbf{1}'} = (x'y')_{\mathbf{1}}.
\end{aligned} \tag{41}$$

In addition, obviously the tensor product of two trivial singlets corresponds to a trivial singlet.

Tensor products are important to applications for particle phenomenology. Matter and Higgs fields may be assigned to have certain representations of discrete symmetries. The Lagrangian must be invariant under discrete symmetries. That implies that n-point couplings corresponding to a trivial singlet can appear in Lagrangian.

In addition to the above (real) representation of S_3, another representation, i.e. the complex representation, is often used in the literature. Here, we mention about changing representation bases. All permutations of S_3 in Eq. (19) are represented on the reducible triplet (x_1, x_2, x_3) as

$$\begin{aligned}
&\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \\
&\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}.
\end{aligned} \tag{42}$$

We change the representation through the unitary transformation, $U^\dagger g U$, e.g. by using the unitary matrix U_{tribi},

$$U_{\text{tribi}} = \begin{pmatrix} \sqrt{2/3} & 1/\sqrt{3} & 0 \\ -1/\sqrt{6} & 1/\sqrt{3} & -1/\sqrt{2} \\ -1/\sqrt{6} & 1/\sqrt{3} & 1/\sqrt{2} \end{pmatrix}. \tag{43}$$

Then, the six elements of S_3 are written as

$$\begin{aligned}
&\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & -\frac{1}{2} & -\frac{\sqrt{3}}{2} \\ 0 & -\frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix}, \\
&\begin{pmatrix} 1 & 0 & 0 \\ 0 & -\frac{1}{2} & -\frac{\sqrt{3}}{2} \\ 0 & \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & -\frac{1}{2} & \frac{\sqrt{3}}{2} \\ 0 & \frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & -\frac{1}{2} & \frac{\sqrt{3}}{2} \\ 0 & -\frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix}.
\end{aligned} \tag{44}$$

Note that this form is completely reducible and that the (right-bottom) (2×2) submatrices are exactly the same as those for the doublet representation (33). The unitary matrix $U_{\rm tribi}$ is called the tri-bimaximal matrix and plays a role in the neutrino physics as studied in section 15.

We can use another unitary matrix U in order to obtain a completely reducible form from the reducible representation matrices (42). For example, let us use the following matrix,

$$U_w = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 1 & 1 \\ 1 & w & w^2 \\ 1 & w^2 & w \end{pmatrix}. \tag{45}$$

Then, the six elements of S_3 are written as

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & w^2 \\ 0 & w & 0 \end{pmatrix},$$
$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & w & 0 \\ 0 & 0 & w^2 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & w \\ 0 & w^2 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & w^2 & 0 \\ 0 & 0 & w \end{pmatrix}. \tag{46}$$

The (right-bottom) (2×2) submatrices correspond to the doublet representation in the different basis, that is, the complex representation. This unitary matrix is called the magic matrix. In different bases, the multiplication rule does not change. For example, we obtain $\mathbf{2} \times \mathbf{2} = \mathbf{1} + \mathbf{1}' + \mathbf{2}$ in both the real and complex bases. However, elements of doublets in the left hand side are written in a different way.

3.2 S_4

S_4 consists of all permutations among four objects, (x_1, x_2, x_3, x_4),

$$(x_1, x_3, x_2, x_4), \quad \rightarrow \quad (x_i, x_j, x_k, x_l), \tag{47}$$

and the order of S_4 is equal to $4! = 24$. We denote all of S_4 elements as

$$\begin{aligned}
&a_1 : (x_1, x_3, x_2, x_4),\ a_2 : (x_2, x_1, x_4, x_3),\ a_3 : (x_3, x_4, x_1, x_2),\ a_4 : (x_4, x_3, x_2, x_1), \\
&b_1 : (x_1, x_4, x_2, x_3),\ b_2 : (x_4, x_1, x_3, x_2),\ b_3 : (x_2, x_3, x_1, x_4),\ b_4 : (x_3, x_2, x_4, x_1), \\
&c_1 : (x_1, x_3, x_4, x_2),\ c_2 : (x_3, x_1, x_2, x_4),\ c_3 : (x_4, x_2, x_1, x_3),\ c_4 : (x_2, x_4, x_3, x_1), \\
&d_1 : (x_1, x_2, x_4, x_3),\ d_2 : (x_2, x_1, x_3, x_4),\ d_3 : (x_4, x_3, x_1, x_2),\ d_4 : (x_3, x_4, x_2, x_1), \\
&e_1 : (x_1, x_3, x_2, x_4),\ e_2 : (x_3, x_1, x_4, x_2),\ e_3 : (x_2, x_4, x_1, x_3),\ e_4 : (x_4, x_2, x_3, x_1), \\
&f_1 : (x_1, x_4, x_3, x_2),\ f_2 : (x_4, x_1, x_2, x_3),\ f_3 : (x_3, x_2, x_1, x_4),\ f_4 : (x_2, x_3, x_4, x_1),
\end{aligned} \tag{48}$$

where we have shown the ordering of four objects after permutations. The S_4 is a symmetry of a cube as shown in Figure 2.

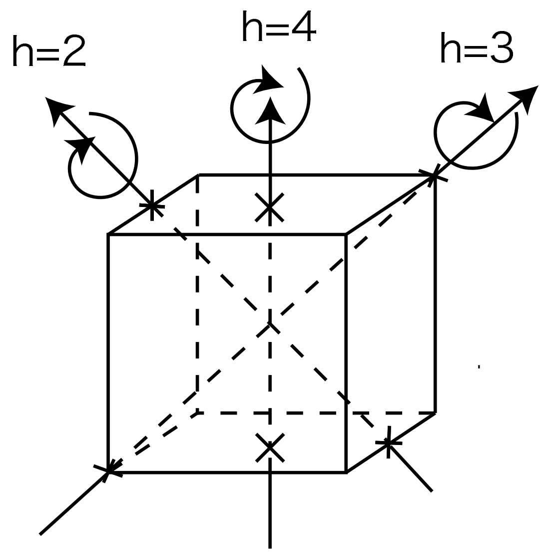

Figure 2: The S_4 symmetry of a cube. This figure shows the transformations corresponding to the S_4 elements with $h = 2, 3$ and 4. Note that the group can be also considered as the regular octahedron in a way similar to a cube.

It is obvious that $x_1 + x_2 + x_3 + x_4$ is invariant under any permutation of S_4, that is, a trivial singlet. Thus, we use the vector space, which is orthogonal to this singlet direction,

$$\mathbf{3} : \begin{pmatrix} A_x \\ A_y \\ A_z \end{pmatrix} = \begin{pmatrix} x_1 + x_2 - x_3 - x_4 \\ x_1 - x_2 + x_3 - x_4 \\ x_1 - x_2 - x_3 + x_4 \end{pmatrix}, \tag{49}$$

in order to construct matrix representations of S_4, that is, a triplet representation. In this triplet vector space, all of S_4 elements are represented by the following matrices,

$$
\begin{aligned}
a_1 &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad a_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \\
a_3 &= \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \quad a_4 = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \\
b_1 &= \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \quad b_2 = \begin{pmatrix} 0 & 0 & 1 \\ -1 & 0 & 0 \\ 0 & -1 & 0 \end{pmatrix}, \\
b_3 &= \begin{pmatrix} 0 & 0 & -1 \\ 1 & 0 & 0 \\ 0 & -1 & 0 \end{pmatrix}, \quad b_4 = \begin{pmatrix} 0 & 0 & -1 \\ -1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \\
c_1 &= \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, \quad c_2 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & -1 \\ -1 & 0 & 0 \end{pmatrix}, \\
c_3 &= \begin{pmatrix} 0 & -1 & 0 \\ 0 & 0 & 1 \\ -1 & 0 & 0 \end{pmatrix}, \quad c_4 = \begin{pmatrix} 0 & -1 & 0 \\ 0 & 0 & -1 \\ 1 & 0 & 0 \end{pmatrix}, \\
d_1 &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad d_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{pmatrix}, \\
d_3 &= \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}, \quad d_4 = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}, \\
e_1 &= \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad e_2 \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \\
e_3 &= \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \quad e_4 = \begin{pmatrix} 0 & -1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \\
f_1 &= \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad f_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \\ -1 & 0 & 0 \end{pmatrix}, \\
f_3 &= \begin{pmatrix} 0 & 0 & -1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{pmatrix}, \quad f_4 = \begin{pmatrix} 0 & 0 & -1 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{pmatrix}.
\end{aligned} \tag{50}
$$

- **Conjugacy classes**

	h	χ_1	$\chi_{1'}$	χ_2	χ_3	$\chi_{3'}$
C_1	1	1	1	2	3	3
C_3	2	1	1	2	-1	-1
C_6	2	1	-1	0	1	-1
$C_{6'}$	4	1	-1	0	-1	1
C_8	3	1	1	-1	0	0

Table 2: Characters of S_4 representations

The S_4 elements can be classified by the order h of each element, where $a^h = e$, as

$$\begin{array}{lcc} h = 1 & : & \{a_1\}, \\ h = 2 & : & \{a_2, a_3, a_4, d_1, d_2, e_1, e_4, f_1, f_3\}, \\ h = 3 & : & \{b_1, b_2, b_3, b_4, c_1, c_2, c_3, c_4\}, \\ h = 4 & : & \{d_3, d_4, e_2, e_3, f_2, f_4\}. \end{array} \tag{51}$$

Moreover, they are classified by the conjugacy classes as

$$\begin{array}{rccl} C_1 & : & \{a_1\}, & h = 1, \\ C_3 & : & \{a_2, a_3, a_4\}, & h = 2, \\ C_6 & : & \{d_1, d_2, e_1, e_4, f_1, f_3\}, & h = 2, \\ C_8 & : & \{b_1, b_2, b_3, b_4, c_1, c_2, c_3, c_4\}, & h = 3, \\ C_{6'} & : & \{d_3, d_4, e_2, e_3, f_2, f_4\}, & h = 4. \end{array} \tag{52}$$

• **Characters and representations**

Thus, S_4 includes five conjugacy classes, that is, there are five irreducible representations. For example, all of elements are written as multiplications of b_1 in C_8 and d_4 in $C_{6'}$, which satisfy

$$(b_1)^3 = e, \quad (d_4)^4 = e, \quad d_4(b_1)^2 d_4 = b_1, \quad d_4 b_1 d_4 = b_1 (d_4)^2 b_1. \tag{53}$$

The orthogonality relation (12) requires

$$\sum_\alpha [\chi_\alpha(C_1)]^2 = \sum_n m_n n^2 = m_1 + 4m_2 + 9m_3 + \cdots = 24, \tag{54}$$

like Eq. (25), and m_n also satisfy $m_1 + m_2 + m_3 + \cdots = 5$, because there must be five irreducible representations. Then, their unique solution is obtained as $(m_1, m_2, m_3) = (2, 1, 2)$. That is, irreducible representations of S_4 include two singlets $\mathbf{1}$ and $\mathbf{1}'$, one doublet $\mathbf{2}$, and two triplets $\mathbf{3}$ and $\mathbf{3}'$, where $\mathbf{1}$ corresponds to a trivial singlet and $\mathbf{3}$ corresponds to (49) and (50). We can compute the character for each representation by an analysis similar to S_3 in the previous subsection. The results are shown in Table 2.

For $\mathbf{2}$, the representation matrices are written as e.g.

$$a_2(\mathbf{2}) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \ b_1(\mathbf{2}) = \begin{pmatrix} \omega & 0 \\ 0 & \omega^2 \end{pmatrix},$$
$$d_1(\mathbf{2}) = d_3(\mathbf{2}) = d_4(\mathbf{2}) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \tag{55}$$

For $\mathbf{3}'$, the representation matrices are written as e.g.

$$a_2(\mathbf{3}') = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \ b_1(\mathbf{3}') = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \tag{56}$$
$$d_1(\mathbf{3}') = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{pmatrix}, \ d_3(\mathbf{3}') = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}, \ d_4(\mathbf{3}') = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}.$$

Note that $a_2(\mathbf{3}') = a_2(\mathbf{3})$ and $b_1(\mathbf{3}') = b_1(\mathbf{3})$, but $d_1(\mathbf{3}') = -d_1(\mathbf{3})$, $d_3(\mathbf{3}') = -d_3(\mathbf{3})$ and $d_4(\mathbf{3}') = -d_4(\mathbf{3})$. This aspect would be obvious from the above character table.

• Tensor products

Finally, we show the tensor products. The tensor products of $\mathbf{3}\times\mathbf{3}$ can be decomposed as

$$(\mathbf{A})_{\mathbf{3}} \times (\mathbf{B})_{\mathbf{3}} = (\mathbf{A}\cdot\mathbf{B})_{\mathbf{1}} + \begin{pmatrix} \mathbf{A}\cdot\Sigma\cdot\mathbf{B} \\ \mathbf{A}\cdot\Sigma^*\cdot\mathbf{B} \end{pmatrix}_{\mathbf{2}} + \begin{pmatrix} \{A_yB_z\} \\ \{A_zB_x\} \\ \{A_xB_y\} \end{pmatrix}_{\mathbf{3}} + \begin{pmatrix} [A_yB_z] \\ [A_zB_x] \\ [A_xB_y] \end{pmatrix}_{\mathbf{3}'}, \tag{57}$$

where

$$\begin{aligned} \mathbf{A}\cdot\mathbf{B} &= A_xB_x + A_yB_y + A_zB_z, \\ \{A_iB_j\} &= A_iB_j + A_jB_i, \\ [A_yB_z] &= A_iB_j - A_jB_i, \\ \mathbf{A}\cdot\Sigma\cdot\mathbf{B} &= A_xB_x + \omega A_yB_y + \omega^2 A_zB_z, \\ \mathbf{A}\cdot\Sigma^*\cdot\mathbf{B} &= A_xB_x + \omega^2 A_yB_y + \omega A_zB_z. \end{aligned} \tag{58}$$

The tensor products of other representations are also decomposed as e.g.

$$(\mathbf{A})_{\mathbf{3}'} \times (\mathbf{B})_{\mathbf{3}'} = (\mathbf{A}\cdot\mathbf{B})_{\mathbf{1}} + \begin{pmatrix} \mathbf{A}\cdot\Sigma\cdot\mathbf{B} \\ \mathbf{A}\cdot\Sigma^*\cdot\mathbf{B} \end{pmatrix}_{\mathbf{2}} + \begin{pmatrix} \{A_yB_z\} \\ \{A_zB_x\} \\ \{A_xB_y\} \end{pmatrix}_{\mathbf{3}} + \begin{pmatrix} [A_yB_z] \\ [A_zB_x] \\ [A_xB_y] \end{pmatrix}_{\mathbf{3}'}, \tag{59}$$
$$(\mathbf{A})_{\mathbf{3}} \times (\mathbf{B})_{\mathbf{3}'} = (\mathbf{A}\cdot\mathbf{B})_{\mathbf{1}'} + \begin{pmatrix} \mathbf{A}\cdot\Sigma\cdot\mathbf{B} \\ -\mathbf{A}\cdot\Sigma^*\cdot\mathbf{B} \end{pmatrix}_{\mathbf{2}} + \begin{pmatrix} \{A_yB_z\} \\ \{A_zB_x\} \\ \{A_xB_y\} \end{pmatrix}_{\mathbf{3}'} + \begin{pmatrix} [A_yB_z] \\ [A_zB_x] \\ [A_xB_y] \end{pmatrix}_{\mathbf{3}}, \tag{60}$$

and

$$(\mathbf{A})_{\mathbf{2}} \times (\mathbf{B})_{\mathbf{2}} = \{A_x B_y\}_{\mathbf{1}} + [A_x B_y]_{\mathbf{1}'} + \begin{pmatrix} A_y B_y \\ A_x B_x \end{pmatrix}_{\mathbf{2}}, \tag{61}$$

$$\begin{pmatrix} A_x \\ A_y \end{pmatrix}_{\mathbf{2}} \times \begin{pmatrix} B_x \\ B_y \\ B_z \end{pmatrix}_{\mathbf{3}} = \begin{pmatrix} (A_x + A_y)B_x \\ (\omega^2 A_x + \omega A_y)B_y \\ (\omega A_x + \omega^2 A_y)B_z \end{pmatrix}_{\mathbf{3}} + \begin{pmatrix} (A_x - A_y)B_x \\ (\omega^2 A_x - \omega A_y)B_y \\ (\omega A_x - \omega^2 A_y)B_z \end{pmatrix}_{\mathbf{3}'}, \tag{62}$$

$$\begin{pmatrix} A_x \\ A_y \end{pmatrix}_{\mathbf{2}} \times \begin{pmatrix} B_x \\ B_y \\ B_z \end{pmatrix}_{\mathbf{3}'} = \begin{pmatrix} (A_x + A_y)B_x \\ (\omega^2 A_x + \omega A_y)B_y \\ (\omega A_x + \omega^2 A_y)B_z \end{pmatrix}_{\mathbf{3}'} + \begin{pmatrix} (A_x - A_y)B_x \\ (\omega^2 A_x - \omega A_y)B_y \\ (\omega A_x - \omega^2 A_y)B_z \end{pmatrix}_{\mathbf{3}}. \tag{63}$$

Furthermore, we have $\mathbf{3} \times \mathbf{1}' = \mathbf{3}'$ and $\mathbf{3}' \times \mathbf{1}' = \mathbf{3}$ and $\mathbf{2} \times \mathbf{1}' = \mathbf{2}$.

In the literature, several bases are used for S_4. The decomposition of tensor products, $\mathbf{r} \times \mathbf{r}' = \sum_m \mathbf{r}_m$, does not depend on the basis. For example, we obtain $\mathbf{3} \times \mathbf{3}' = \mathbf{1}' + \mathbf{2} + \mathbf{3} + \mathbf{3}'$ in any basis. However, the multiplication rules written by components depend on the basis, which we use. We have used the basis (55). In appendix B, we show the relations between several bases and give explicitly the multiplication rules in terms of components.

Similarly, we can study the S_N group with $N > 4$. Here we give a brief comment on such groups. The S_N group with $N > 4$ has only one invariant subgroup, that is, the alternating group, A_N. The S_N group has two one-dimensional representations: one is trivial singlet, that is, invariant under all the elements (symmetric representation), the other is pseudo singlet, that is, symmetric under the even permutation-elements but antisymmetric under the odd permutation-elements. Group-theoretical aspects of S_5 are derived from those of S_4 by applying a theorem of Frobenuis(Frobenuis formula), graphical method(Young tableaux), recursion formulas for characters (branching laws). The details are given in, e.g., the text book of [24]. Such analysis would be extended recursively from S_N to S_{N+1}.

4 A_N

All even permutations among S_N form a group, which is called A_N with the order $(N!)/2$. It is often called the alternating group. For example, among S_3 in subsection 3.1 the even permutations include

$$\begin{aligned} e &: (x_1, x_2, x_3) \to (x_1, x_2, x_3), \\ a_4 &: (x_1, x_2, x_3) \to (x_3, x_1, x_2), \\ a_5 &: (x_1, x_2, x_3) \to (x_2, x_3, x_1), \end{aligned} \tag{64}$$

while the odd permutations include

$$\begin{aligned} a_1 &: (x_1, x_2, x_3) \to (x_2, x_1, x_3), \\ a_2 &: (x_1, x_2, x_3) \to (x_3, x_2, x_1), \\ a_3 &: (x_1, x_2, x_3) \to (x_1, x_3, x_2). \end{aligned} \tag{65}$$

The three elements, $\{e, a_4, a_5\}$ form the group A_3. Since $(a_4)^2 = a_5$ and $(a_4)^3 = e$, the group A_3 is nothing but Z_3. Thus, the smallest non-Abelian group is A_4.

4.1 A_4

All even permutations of S_4 form A_4, whose order is equal to $(4!)/2 = 12$. The A_4 group is the symmetry of a tetrahedron as shown in Figure 3. Thus, the A_4 group is often denoted as T. Using the notation in subsection 3.2, all of 12 elements are denoted as

$$\begin{aligned}
a_1 &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad a_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \\
a_3 &= \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \quad a_4 = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \\
b_1 &= \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \quad b_2 = \begin{pmatrix} 0 & 0 & 1 \\ -1 & 0 & 0 \\ 0 & -1 & 0 \end{pmatrix}, \\
b_3 &= \begin{pmatrix} 0 & 0 & -1 \\ 1 & 0 & 0 \\ 0 & -1 & 0 \end{pmatrix}, \quad b_4 = \begin{pmatrix} 0 & 0 & -1 \\ -1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \\
c_1 &= \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, \quad c_2 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & -1 \\ -1 & 0 & 0 \end{pmatrix}, \\
c_3 &= \begin{pmatrix} 0 & -1 & 0 \\ 0 & 0 & 1 \\ -1 & 0 & 0 \end{pmatrix}, \quad c_4 = \begin{pmatrix} 0 & -1 & 0 \\ 0 & 0 & -1 \\ 1 & 0 & 0 \end{pmatrix}.
\end{aligned} \tag{66}$$

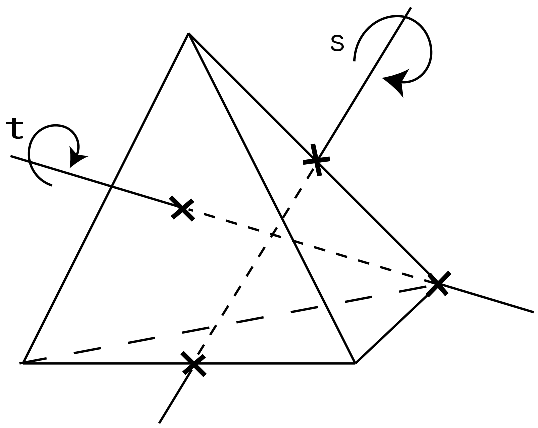

Figure 3: The A_4 symmetry of tetrahedron.

From these forms, it is found obviously that A_4 is isomorphic to $\Delta(12) \simeq (Z_2 \times Z_2) \rtimes Z_3$, which is explained in section 9.

They are classified by the conjugacy classes as

$$\begin{array}{cccc} C_1 & : & \{a_1\}, & h=1, \\ C_3 & : & \{a_2, a_3, a_4\}, & h=2, \\ C_4 & : & \{b_1, b_2, b_3, b_4,\}, & h=3, \\ C_{4'} & : & \{c_1, c_2, c_3, c_4,\}, & h=3, \end{array} \tag{67}$$

where we have also shown the orders of each element in the conjugacy class by h. There are four conjugacy classes and there must be four irreducible representations, i.e. $m_1 + m_2 + m_3 + \cdots = 4$.

The orthogonality relation (11) requires

$$\sum_\alpha [\chi_\alpha(C_1)]^2 = \sum_n m_n n^2 = m_1 + 4m_2 + 9m_3 + \cdots = 12, \tag{68}$$

for m_i, which satisfy $m_1 + m_2 + m_3 + \cdots = 4$. The solution is obtained as $(m_1, m_2, m_3) = (3, 0, 1)$. That is, the A_4 group has three singlets, $\mathbf{1}$, $\mathbf{1}'$, and $\mathbf{1}''$, and a single triplet $\mathbf{3}$, where the triplet corresponds to (66).

Another algebraic definition of A_4 is often used in the literature. We denote $a_1 = e$, $a_2 = s$ and $b_1 = t$. They satisfy the following algebraic relations,

$$s^2 = t^3 = (st)^3 = e. \tag{69}$$

The closed algebra of these elements, s and t, is defined as the A_4. It is straightforward to write all of a_i, b_i and c_i elements by s and t. Then, the conjugacy classes are rewritten

	h	χ_1	$\chi_{1'}$	$\chi_{1''}$	χ_3
C_1	1	1	1	1	3
C_3	2	1	1	1	-1
C_4	3	1	ω	ω^2	0
$C_{4'}$	3	1	ω^2	ω	0

Table 3: Characters of A_4 representations

as

$$\begin{array}{rcc} C_1: & \{e\}, & h=1, \\ C_3: & \{s, tst^2, t^2st\}, & h=2, \\ C_4: & \{t, ts, st, sts\}, & h=3, \\ C_{4'}: & \{t^2, st^2, t^2s, tst\}, & h=3. \end{array} \tag{70}$$

Using them, we study characters. First, we consider characters of three singlets. Because $s^2 = e$, the characters of C_3 have two possibilities, $\chi_\alpha(C_3) = \pm 1$. However, the two elements, t and ts, belong to the same conjugacy class C_4. That means that $\chi_\alpha(C_3)$ should have the unique value, $\chi_\alpha(C_3) = 1$. Similarly, because of $t^3 = e$, the characters $\chi_\alpha(t)$ can correspond to three values, i.e. $\chi_\alpha(t) = \omega^n$, $n = 0, 1, 2$, and all of these three values are consistent with the above structure of conjugacy classes. Thus, all of three singlets, $\mathbf{1}$, $\mathbf{1}'$ and $\mathbf{1}''$ are classified by these three values of $\chi_\alpha(t) = 1, \omega$ and ω^2, respectively. Obviously, it is found that $\chi_\alpha(C_{4'}) = (\chi_\alpha(C_4))^2$. Thus, the generators such as $s = a_2, t = b_1, t^2 = c_1$ are represented on the non-trivial singlets $\mathbf{1}'$ and $\mathbf{1}''$ as

$$\begin{array}{lll} s(\mathbf{1}') = a_2(\mathbf{1}') = 1, & t(\mathbf{1}') = b_1(\mathbf{1}') = \omega, & t^2(\mathbf{1}') = c_1(\mathbf{1}') = \omega^2, \\ s(\mathbf{1}'') = a_2(\mathbf{1}'') = 1, & t(\mathbf{1}'') = b_1(\mathbf{1}'') = \omega^2, & t^2(\mathbf{1}'') = c_1(\mathbf{1}'') = \omega. \end{array} \tag{71}$$

These characters are shown in Table 3. Next, we consider the characters for the triplet representation. Obviously, the matrices in Eq. (66) correspond to the triplet representation. Thus, we can obtain their characters. Its result is also shown in Table 3.

The tensor product of $\mathbf{3} \times \mathbf{3}$ can be decomposed as

$$\begin{aligned} (\mathbf{A})_\mathbf{3} \times (\mathbf{B})_\mathbf{3} &= (\mathbf{A} \cdot \mathbf{B})_\mathbf{1} + (\mathbf{A} \cdot \Sigma \cdot \mathbf{B})_{\mathbf{1}'} + (\mathbf{A} \cdot \Sigma^* \cdot \mathbf{B})_{\mathbf{1}''} \\ &\quad + \begin{pmatrix} \{A_y B_z\} \\ \{A_z B_x\} \\ \{A_x B_y\} \end{pmatrix}_\mathbf{3} + \begin{pmatrix} [A_y B_z] \\ [A_z B_x] \\ [A_x B_y] \end{pmatrix}_\mathbf{3} . \end{aligned} \tag{72}$$

4.2 A_5

Here, we mention briefly about the A_5 group. The A_5 group is isomorphic to the symmetry of a regular icosahedron. Thus, it is pedagogical to explain group-theoretical aspects of A_5 as the symmetry of a regular icosahedron [152]. As shown in Figure 4, a regular

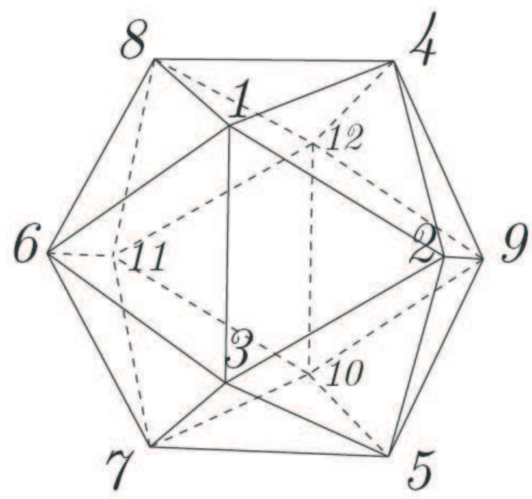

Figure 4: The regular icosahedron.

icosahedron consists of 20 identical equilateral triangular faces, 30 edges and 12 vertices. The icosahedron is dual to a dodecahedron, whose symmetry is also isomorphic to A_5. The A_5 elements correspond to all the proper rotations of the icosahedron. Such rotations are classified into five types, that is, the 0 rotation (identity), π rotations about the midpoint of each edge, rotations by $2\pi/3$ about axes through the center of each face, and rotations by $2\pi/5$ and $4\pi/5$ about an axis through each vertex. Following [152], we label the vertex by number $n = 1, \cdots, 12$ in Figure 4. Here, we define two elements a and b such that a corresponds to the rotation by π about the midpoint of the edge between vertices 1 and 2 while b corresponds the rotation by $2\pi/3$ about the axis through the center of the triangular face 10-11-12. That is, these two elements correspond to the transformations acting on the 12 vertices as follows,

$$
\begin{aligned}
a &: (1,2,3,4,5,6,7,8,9,10,11,12) \to (2,1,4,3,8,9,12,5,6,11,10,7), \\
b &: (1,2,3,4,5,6,7,8,9,10,11,12) \to (2,3,1,5,6,4,8,9,7,11,12,10).
\end{aligned}
$$

Then the product ab is given by the following transformation as

$$
ab : (1,2,3,4,5,6,7,8,9,10,11,12) \to (3,2,5,1,9,7,10,6,4,12,11,8),
$$

which is the rotation by $2\pi/5$ about the axis through the vertex 2. All of the A_5 elements are written by products of these elements, which satisfy

$$
a^2 = b^3 = (ab)^5 = e. \tag{73}
$$

The order of A_5 is equal to $(5!)/2 = 60$. All of the A_5 elements, i.e. all the rotations of the icosahedron, are classified into five conjugacy classes as follows,

$$
\begin{aligned}
C_1 &: \{e\}, \\
C_{15} &: \{a(12), a(13), a(14), a(16), a(18), a(23), a(24), a(25), a(29), a(35)\}, \\
&\quad\ a(36), a(37), a(48), a(49), a(59), \}, \\
C_{20} &: \{b(123), b(124), b(126), b(136), b(168), b(235), b(249), \\
&\quad\ b(259), b(357), b(367), \text{ and their inverse elements}\}, \\
C_{12} &: \{c(1), c(2), c(3), c(4), c(5), c(6), \text{ and their inverse elements}\}, \\
C'_{12} &: \{c^2(1), c^2(2), c^2(3), c^2(4), c^2(5), c^2(6), \text{ and their inverse elements}\},
\end{aligned} \tag{74}
$$

where $a(km)$, $b(kmn)$ and $c(k)$ denote the rotation by π about the midpoint of the edge $k-m$, the rotation by $2\pi/3$ about the axis through the center of the face $k-m-n$ and the rotation by $2\pi/5$ about the axis through the vertex k. The conjugacy classes, C_1, C_{15}, C_{20}, C_{12} and C'_{12}, include 1, 15, 20, 12 and 12 elements, respectively. Since obviously $(a(km))^2 = (b(kmn))^3 = (c(k))^5 = e$, we find $h=2$ in C_{15}, $h=3$ in C_{20}, $h=5$ in C_{12} and $h=5$ in C'_{12}, where h denotes the order of each element in the conjugacy class, i.e. $g^h = e$. The orthogonality relations (12) and (13) for A_5 lead to

$$m_1 + 4m_2 + 9m_3 + 16m_4 + 25m_5 + \cdots = 60, \tag{75}$$

$$m_1 + m_2 + m_3 + m_4 + m_5 + \cdots = 5. \tag{76}$$

The solution is found as $(m_1, m_2, m_3, m_4, m_5) = (1, 0, 2, 1, 1)$. Therefore the A_5 group has one trivial singlet, **1**, two triplets, **3** and **3′**, one quartet, **4**, and one quintet, **5**. The characters are shown in Table 4. Instead of a and b, we use the generators, $s = a$ and $t = bab$, which satisfy

$$s^2 = t^5 = (t^2st^3st^{-1}stst^{-1})^2 = e. \tag{77}$$

The generators, s and t, are represented as [152],

$$s = \frac{1}{2}\begin{pmatrix} -1 & \phi & \frac{1}{\phi} \\ \phi & \frac{1}{\phi} & 1 \\ \frac{1}{\phi} & 1 & -\phi \end{pmatrix}, \quad t = \frac{1}{2}\begin{pmatrix} 1 & \phi & \frac{1}{\phi} \\ -\phi & \frac{1}{\phi} & 1 \\ \frac{1}{\phi} & -1 & \phi \end{pmatrix}, \qquad \text{on } \mathbf{3}, \tag{78}$$

$$s = \frac{1}{2}\begin{pmatrix} -\phi & \frac{1}{\phi} & 1 \\ \frac{1}{\phi} & -1 & \phi \\ 1 & \phi & \frac{1}{\phi} \end{pmatrix}, \quad t = \frac{1}{2}\begin{pmatrix} -\phi & -\frac{1}{\phi} & 1 \\ \frac{1}{\phi} & 1 & \phi \\ -1 & \phi & -\frac{1}{\phi} \end{pmatrix}, \qquad \text{on } \mathbf{3}', \tag{79}$$

$$s = \frac{1}{4}\begin{pmatrix} -1 & -1 & -3 & -\sqrt{5} \\ -1 & 3 & 1 & -\sqrt{5} \\ -3 & 1 & -1 & \sqrt{5} \\ -\sqrt{5} & -\sqrt{5} & \sqrt{5} & -1 \end{pmatrix},$$

$$t = \frac{1}{4}\begin{pmatrix} -1 & 1 & -3 & \sqrt{5} \\ -1 & -3 & 1 & \sqrt{5} \\ 3 & 1 & 1 & \sqrt{5} \\ \sqrt{5} & -\sqrt{5} & -\sqrt{5} & -1 \end{pmatrix}, \qquad \text{on } \mathbf{4}, \tag{80}$$

	h	1	3	$3'$	4	5
C_1	1	1	3	3	4	5
C_{15}	2	1	-1	-1	0	1
C_{20}	3	1	0	0	1	-1
C_{12}	5	1	ϕ	$1-\phi$	-1	0
$C_{12'}$	5	1	$1-\phi$	ϕ	-1	0

Table 4: Characters of A_5 representations, where $\phi = \frac{1+\sqrt{5}}{2}$.

$\mathbf{3} \otimes \mathbf{3} = \mathbf{1} \oplus \mathbf{3} \oplus \mathbf{5}$
$\mathbf{3}' \otimes \mathbf{3}' = \mathbf{1} \oplus \mathbf{3}' \oplus \mathbf{5}$
$\mathbf{3} \otimes \mathbf{3}' = \mathbf{4} \oplus \mathbf{5}$
$\mathbf{3} \otimes \mathbf{4} = \mathbf{3}' \oplus \mathbf{4} \oplus \mathbf{5}$
$\mathbf{3}' \otimes \mathbf{4} = \mathbf{3} \oplus \mathbf{4} \oplus \mathbf{5}$
$\mathbf{3} \otimes \mathbf{5} = \mathbf{3} \oplus \mathbf{3}' \oplus \mathbf{4} \oplus \mathbf{5}$
$\mathbf{3}' \otimes \mathbf{5} = \mathbf{3} \oplus \mathbf{3}' \oplus \mathbf{4} \oplus \mathbf{5}$
$\mathbf{4} \otimes \mathbf{4} = \mathbf{1} \oplus \mathbf{3} \oplus \mathbf{3}' \oplus \mathbf{4} \oplus \mathbf{5}$
$\mathbf{4} \otimes \mathbf{5} = \mathbf{3} \oplus \mathbf{3}' \oplus \mathbf{4} \oplus \mathbf{5} \oplus \mathbf{5}$
$\mathbf{5} \otimes \mathbf{5} = \mathbf{1} \oplus \mathbf{3} \oplus \mathbf{3}' \oplus \mathbf{4} \oplus \mathbf{4} \oplus \mathbf{5} \oplus \mathbf{5}$

Table 5: Multiplication rules for the A_5 group.

$$
s = \frac{1}{2}\begin{pmatrix} \frac{1-3\phi}{4} & \frac{\phi^2}{2} & -\frac{1}{2\phi^2} & \frac{\sqrt{5}}{2} & \frac{\sqrt{3}}{4\phi} \\ \frac{\phi^2}{2} & 1 & 1 & 0 & \frac{\sqrt{3}}{2\phi} \\ -\frac{1}{2\phi^2} & 1 & 0 & -1 & -\frac{\sqrt{3}\phi}{2} \\ \frac{\sqrt{5}}{2} & 0 & -1 & 1 & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{4\phi} & \frac{\sqrt{3}}{2\phi} & -\frac{\sqrt{3}\phi}{2} & -\frac{\sqrt{3}}{2} & \frac{3\phi-1}{4} \end{pmatrix},
$$

$$
t = \frac{1}{2}\begin{pmatrix} \frac{1-3\phi}{4} & -\frac{\phi^2}{2} & -\frac{1}{2\phi^2} & -\frac{\sqrt{5}}{2} & \frac{\sqrt{3}}{4\phi} \\ \frac{\phi^2}{2} & -1 & 1 & 0 & \frac{\sqrt{3}}{2\phi} \\ \frac{1}{2\phi^2} & 1 & 0 & -1 & \frac{\sqrt{3}\phi}{2} \\ -\frac{\sqrt{5}}{2} & 0 & 1 & 1 & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{4\phi} & -\frac{\sqrt{3}}{2\phi} & -\frac{\sqrt{3}\phi}{2} & \frac{\sqrt{3}}{2} & \frac{3\phi-1}{4} \end{pmatrix}, \qquad \text{on } \mathbf{5}, \qquad (81)
$$

where $\phi = \frac{1+\sqrt{5}}{2}$. Furthermore, the multiplication rules are also shown in Table 5.

Here we show some parts of tensor products [155],

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}_{\mathbf{3}} \otimes \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}_{\mathbf{3}} = (x_1y_1 + x_2y_2 + x_3y_3)_{\mathbf{1}} \oplus \begin{pmatrix} x_3y_2 - x_2y_3 \\ x_1y_3 - x_3y_1 \\ x_2y_1 - x_1y_2 \end{pmatrix}_{\mathbf{3}} \oplus \begin{pmatrix} x_2y_2 - x_1y_1 \\ x_2y_1 + x_1y_2 \\ x_3y_2 + x_2y_3 \\ x_1y_3 + x_3y_1 \\ -\frac{1}{\sqrt{3}}(x_1y_1 + x_2y_2 - 2x_3y_3) \end{pmatrix}_{\mathbf{5}}, \tag{82}$$

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}_{\mathbf{3'}} \otimes \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}_{\mathbf{3'}} = (x_1y_1 + x_2y_2 + x_3y_3)_{\mathbf{1}} \oplus \begin{pmatrix} x_3y_2 - x_2y_3 \\ x_1y_3 - x_3y_1 \\ x_2y_1 - x_1y_2 \end{pmatrix}_{\mathbf{3'}} \oplus \begin{pmatrix} \frac{1}{2}(-\frac{1}{\phi}x_1y_1 - \phi x_2y_2 + \sqrt{5}x_3y_3) \\ x_2y_1 + x_1y_2 \\ -(x_3y_1 + x_1y_3) \\ x_2y_3 + x_3y_2 \\ \frac{1}{2\sqrt{3}}((1-3\phi)x_1y_1 + (3\phi-2)x_2y_2 + x_3y_3) \end{pmatrix}_{\mathbf{5}}, \tag{83}$$

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}_{\mathbf{3}} \otimes \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}_{\mathbf{3'}} = \begin{pmatrix} \frac{1}{\phi}x_3y_2 - \phi x_1y_3 \\ \phi x_3y_1 + \frac{1}{\phi}x_2y_3 \\ -\frac{1}{\phi}x_1y_1 + \phi x_2y_2 \\ x_2y_1 - x_1y_2 + x_3y_3 \end{pmatrix}_{\mathbf{4}} \oplus \begin{pmatrix} \frac{1}{2}(\phi^2 x_2y_1 + \frac{1}{\phi^2}x_1y_2 - \sqrt{5}x_3y_3) \\ -(\phi x_1y_1 + \frac{1}{\phi}x_2y_2) \\ \frac{1}{\phi}x_3y_1 - \phi x_2y_3 \\ \phi x_3y_2 + \frac{1}{\phi}x_1y_3 \\ \frac{\sqrt{3}}{2}(\frac{1}{\phi}x_2y_1 + \phi x_1y_2 + x_3y_3) \end{pmatrix}_{\mathbf{5}}. \tag{84}$$

For $\mathbf{4} \otimes \mathbf{4}$ and $\mathbf{5} \otimes \mathbf{5}$, they include the simple form of trivial singlets as

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}_{\mathbf{4}} \otimes \begin{pmatrix} x_1 \\ y_2 \\ y_3 \\ y_4 \end{pmatrix}_{\mathbf{4}} \to (x_1y_1 + x_2y_2 + x_3y_3 + x_4y_4)_{\mathbf{1}} , \tag{85}$$

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix}_{\mathbf{5}} \otimes \begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \\ y_5 \end{pmatrix}_{\mathbf{5}} \to (x_1y_1 + x_2y_2 + x_3y_3 + x_4y_4 + x_5y_5)_{\mathbf{1}} . \tag{86}$$

5 T'

The T' group is the double covering group of $A_4 = T$. Instead of Eq. (69), we consider the following algebraic relations,

$$s^2 = r, \qquad r^2 = t^3 = (st)^3 = e, \qquad rt = tr. \tag{87}$$

The closed algebra including r, s and t is the T' group. It consists of 24 elements.

- **Conjugacy classes**

  All of 24 elements in T' are classified by their orders as

$$\begin{array}{lcc} h=1 & : & \{e\}, \\ h=2 & : & \{r\}, \\ h=3 & : & \{t, t^2, ts, st, rst^2, rt^2s, rtst, rsts\}, \\ h=4 & : & \{s, rs, tst^2, t^2st, rtst^2, rt^2st\}, \\ h=6 & : & \{rt, rst, rts, rt^2, sts, st^2, t^2s, tst\}. \end{array} \tag{88}$$

Furthermore, they are classified into seven conjugacy classes as

$$\begin{array}{rcl} C_1 : & \{e\}, & h=1, \\ C_{1'} : & \{r\}, & h=2, \\ C_4 : & \{t, rsts, st, ts\}, & h=3, \\ C_{4'} : & \{t^2, rtst, rt^2s, rst^2\}, & h=3, \\ C_6 : & \{s, rs, tst^2, t^2st, rtst^2, rt^2st\}, & h=4, \\ C_{4''} : & \{rt, sts, rst, rts\}, & h=6, \\ C_{4'''} : & \{rt^2, tst, t^2s, st^2\}, & h=6. \end{array} \tag{89}$$

- **Characters and representations**

  The orthogonality relations (12) and (13) for T' lead to

$$m_1 + 2^2 m_2 + 3^2 m_3 + \cdots = 24, \tag{90}$$

$$m_1 + m_2 + m_3 + \cdots = 7. \tag{91}$$

The solution is found as $(m_1, m_2, m_3) = (3, 3, 1)$. That is, there are three singlets, three doublets and a triplet.

Now, let us study characters. The analysis on T' is quite similar to one on $A_4 = T$. First, we start with singlets. Because of $s^4 = r^2 = e$, there are four possibilities for $\chi_\alpha(s) = (i)^n$ $(n = 0, 1, 2, 3)$. However, since t and ts belong to the same conjugacy class, C_4, the character consistent with the structure of conjugacy classes is obtained as $\chi_\alpha(s) = 1$ for singlets. That also means $\chi_\alpha(r) = 1$. Then, similarly to $A_4 = T$, three singlets are classified by three possible values of $\chi_\alpha(t) = \omega^n$. That is, three singlets, $\mathbf{1}$, $\mathbf{1}'$, and $\mathbf{1}''$ are classified by these three values of $\chi_\alpha(t) = 1, \omega$ and ω^2, respectively. These are shown in Table 6.

Next, let us study three doublet representations **2**, **2′** and **2″**, and a triplet representation **3** for r. The element r commutes with all of elements. That implies by the Shur's lemma that r can be represented by

$$\lambda_{2,2',2''}\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}, \tag{92}$$

on **2**, **2′** and **2″** and

$$\lambda_{3}\begin{pmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix}, \tag{93}$$

on **3**. In addition, possible values of $\lambda_{2,2',2''}$ and λ_3 must be equal to $\lambda_{2,2',2''} = \pm 1$ and $\lambda_3 = \pm 1$ because of $r^2 = e$. That is, possible values of characters are obtained as $\chi_2(r), \chi_{2'}(r), \chi_{2''}(r) = \pm 2$ and $\chi_3(r) = \pm 3$. Furthermore, the second orthogonality relation between e and r leads to

$$\sum_{\alpha}\chi_{D_\alpha}(e)^*\chi_{D_\alpha}(r) = 3 + 2\chi_2(r) + 2\chi_{2'}(r) + 2\chi_{2''}(r) + 3\chi_3(r) = 0. \tag{94}$$

Here we have used $\chi_{1,1',1''}(r) = 1$. Its solution is obtained as $\chi_2(r) = \chi_{2'}(r) = \chi_{2''}(r) = -2$ and $\chi_3(r) = 3$. These are shown in Table 6. That is, the r element is represented by

$$r = -\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}, \tag{95}$$

on **2**, **2′** and **2″**, and

$$r = \begin{pmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix}, \tag{96}$$

on **3**.

Now, let us study doublet representation of t. We use the basis diagonalizing t. Because of $t^3 = e$, the element t could be written as

$$\begin{pmatrix}\omega^k & 0\\ 0 & \omega^\ell\end{pmatrix}, \tag{97}$$

with $k, \ell = 0, 1, 2$. However, if $k = \ell$, the above matrix would become proportional to the (2×2) identity matrix, that is, the element t would also commute with all of elements. That is nothing but a singlet representation. Then, we should have the condition $k \neq \ell$. As a result, there are three possible values for the trace of the above values as $\omega^k + \omega^\ell = -\omega^n$ with $k, \ell, n = 0, 1, 2$ and $k \neq \ell, \ell \neq n, n \neq k$. That is, the characters of t for three doublets, **2**, **2**′ and **2**″ are classified as $\chi_2(t) = -1, \chi_{2'}(t) = -\omega$ and $\chi_{2''}(t) = -\omega^2$. These are shown in Table 6. Then, the element t is represented by

$$t = \begin{pmatrix} \omega^2 & 0 \\ 0 & \omega \end{pmatrix}, \qquad \text{on } \mathbf{2}, \tag{98}$$

$$t = \begin{pmatrix} 1 & 0 \\ 0 & \omega^2 \end{pmatrix}, \qquad \text{on } \mathbf{2}', \tag{99}$$

$$t = \begin{pmatrix} \omega & 0 \\ 0 & 1 \end{pmatrix}, \qquad \text{on } \mathbf{2}''. \tag{100}$$

Since we have found the explicit (2×2) matrices for r and t on all of three doublets, it is straightforward to calculate the explicit forms of rt and rt^2, which belong to the conjugacy classes, C_4' and C_4'' , respectively. Then, it is also straightforward to compute the characters of C_4' and C_4'' for doublets by such explicit forms of (2×2) matrices for rt and rt^2. They are shown in Table 6.

In order to determine the character of t for the triplet, $\chi_3(t)$, we use the second orthogonality relation between e and t,

$$\sum_\alpha \chi_{D_\alpha}(e)^* \chi_{D_\alpha}(t) = 0. \tag{101}$$

Note that all of the characters $\chi_\alpha(t)$ except $\chi_3(t)$ have been derived in the above. Then, the above orthogonality relation (101) requires $\chi_3(t) = 0$, that is, $\chi_3(C_4) = 0$. Similarly, it is found that $\chi_3(C_4') = \chi_3(C_4'') = \chi_3(C_4''') = 0$ as shown in Table 6. Now, we study the explicit form of the (3×3) matrix for t on the triplet. We use the basis to diagonalize t. Since $t^3 = e$ and $\chi_3(t) = 0$, we can obtain

$$t = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \omega & 0 \\ 0 & 0 & \omega^2 \end{pmatrix}, \qquad \text{on } \mathbf{3}. \tag{102}$$

Finally, we study the characters of C_6 including s for the doublets and the triplet. Here, we use the first orthogonality relation between the trivial singlet representation and the doublet representation **2**

$$\sum_{g \in G} \chi_1(g)^* \chi_2(g) = 0. \tag{103}$$

Recall that all of characters except $\chi_2(C_6)$ have been already given. This orthogonality relation (103) requires $\chi_2(C_6) = 0$. Similarly, we find that $\chi_{2'}(C_6) = \chi_{2''}(C_6) = 0$. Furthermore, the character of C_6 for the triplet $\chi_3(C_6)$ is also determined by using the orthogonality relation $\sum_{g \in G} \chi_1(g)^* \chi_2(g) = 0$ with the other known characters. As a result, we obtain $\chi_3(C_6) = -1$. Now, we have completed all of characters in the T' group, which are summarized in Table 6. Then, we study the explicit form of s on the doublets and triplet. On the doublets, the element must be the (2×2) unitary matrix, which satisfies $\mathrm{tr}(s) = 0$ and $s^2 = r$. Recall that the doublet representation for r is already obtained in Eq. (95). Thus, the element s could be represented as

$$s = -\frac{1}{\sqrt{3}} \begin{pmatrix} i & \sqrt{2}p \\ \sqrt{2}\bar{p} & -i \end{pmatrix}, \quad p = i\phi, \tag{104}$$

on the doublet representations. For example, for **2** this representation of s satisfies

$$\mathrm{tr}(st) = -\frac{i}{\sqrt{3}}(\omega^2 - \omega) = -1, \tag{105}$$

so the ambiguity of p cannot be removed.

Here, we summarize the doublet and triplet representations,

$$t = \begin{pmatrix} \omega^2 & 0 \\ 0 & \omega \end{pmatrix}, \quad r = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, \quad s = -\frac{1}{\sqrt{3}} \begin{pmatrix} i & \sqrt{2}p \\ -\sqrt{2}\bar{p} & -i \end{pmatrix} \quad \text{on } \mathbf{2}, \tag{106}$$

$$t = \begin{pmatrix} 1 & 0 \\ 0 & \omega^2 \end{pmatrix}, \quad r = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, \quad s = -\frac{1}{\sqrt{3}} \begin{pmatrix} i & \sqrt{2}p \\ -\sqrt{2}\bar{p} & -i \end{pmatrix} \quad \text{on } \mathbf{2}', \tag{107}$$

$$t = \begin{pmatrix} \omega & 0 \\ 0 & 1 \end{pmatrix}, \quad r = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, \quad s = -\frac{1}{\sqrt{3}} \begin{pmatrix} i & \sqrt{2}p \\ -\sqrt{2}\bar{p} & -i \end{pmatrix} \quad \text{on } \mathbf{2}'', \tag{108}$$

$$t = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \omega & 0 \\ 0 & 0 & \omega^2 \end{pmatrix}, \quad r = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad s = \frac{1}{3} \begin{pmatrix} -1 & 2p_1 & 2p_1p_2 \\ 2\bar{p}_1 & -1 & 2p_2 \\ 2\bar{p}_1\bar{p}_2 & 2\bar{p}_2 & -1 \end{pmatrix} \quad \text{on } \mathbf{3}, \tag{109}$$

where $p_1 = e^{i\phi_1}$ and $p_2 = e^{i\phi_2}$.

• Tensor products

From the above relations, complete tensor products can be determined. First, we study the tensor product of **2** and **2**, i.e.

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}_{\mathbf{2}} \otimes \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}_{\mathbf{2}}. \tag{110}$$

Then, we investigate the transformation property of elements $x_i y_j$ for $i, j = 1, 2$ under t, r and s. Then, it is found that

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}_{\mathbf{2}(\mathbf{2}')} \otimes \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}_{\mathbf{2}(\mathbf{2}'')} = \left(\frac{x_1y_2 - x_2y_1}{\sqrt{2}} \right)_{\mathbf{1}} \oplus \begin{pmatrix} \frac{i}{\sqrt{2}} p_1 p_2 \bar{p} (x_1y_2 + x_2y_1) \\ p_2 \bar{p}^2 x_1 y_1 \\ x_2 y_2 \end{pmatrix}_{\mathbf{3}}. \tag{111}$$

Similarly, we can obtain

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}_{\mathbf{2'(2)}} \otimes \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}_{\mathbf{2'(2'')}} = \left(\frac{x_1y_2 - x_2y_1}{\sqrt{2}}\right)_{\mathbf{1''}} \oplus \begin{pmatrix} p_1\bar{p}^2 x_1 y_1 \\ x_2 y_2 \\ \frac{i}{\sqrt{2}}\bar{p}\bar{p}_2(x_1y_2 + x_2y_1) \end{pmatrix}_{\mathbf{3}}, \tag{112}$$

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}_{\mathbf{2''(2)}} \otimes \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}_{\mathbf{2''(2')}} = \left(\frac{x_1y_2 - x_2y_1}{\sqrt{2}}\right)_{\mathbf{1'}} \oplus \begin{pmatrix} x_2 y_2 \\ \frac{i}{\sqrt{2}}\bar{p}\bar{p}_1(x_1y_2 + x_2y_1) \\ \bar{p}^2\bar{p}_1\bar{p}_2 x_1 y_1 \end{pmatrix}_{\mathbf{3}}. \tag{113}$$

Furthermore, we can compute other products such as $\mathbf{2} \times \mathbf{2'}$, $\mathbf{2} \times \mathbf{2''}$ and $\mathbf{2'} \times \mathbf{2''}$. Then, it is found that

$$\mathbf{2} \times \mathbf{2'} = \mathbf{2''} \times \mathbf{2''}, \qquad \mathbf{2} \times \mathbf{2''} = \mathbf{2'} \times \mathbf{2'}, \qquad \mathbf{2'} \times \mathbf{2''} = \mathbf{2} \times \mathbf{2}. \tag{114}$$

Moreover, a similar analysis leads to

$$\begin{aligned} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}_{\mathbf{3}} \otimes \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}_{\mathbf{3}} &= [x_1y_1 + p_1^2 p_2(x_2y_3 + x_3y_2)]_{\mathbf{1}} \\ &\oplus [x_3y_3 + \bar{p}_1\bar{p}_2^2(x_1y_2 + x_2y_1)]_{\mathbf{1'}} \oplus [(x_2y_2 + \bar{p}_1 p_2(x_1y_3 + x_3y_1)]_{\mathbf{1''}} \\ &\oplus \begin{pmatrix} 2x_1y_1 - p_1^2p_2(x_2y_3 + x_3y_3) \\ 2p_1p_2^2 x_3y_3 - x_1y_2 - x_2y_1 \\ 2p_1\bar{p}_2 x_2y_2 - x_1y_3 - x_3y_1 \end{pmatrix}_{\mathbf{3}} \\ &\oplus \begin{pmatrix} x_2y_3 - x_3y_2 \\ \bar{p}_1^2\bar{p}_2(x_1y_2 - x_2y_1) \\ \bar{p}_1^2\bar{p}_2(x_3y_1 - x_1y_3) \end{pmatrix}_{\mathbf{3}}, \end{aligned} \tag{115}$$

$$\begin{aligned} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}_{\mathbf{2},\mathbf{2'},\mathbf{2''}} \otimes \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}_{\mathbf{3}} &= \begin{pmatrix} -i\sqrt{2}pp_1x_2y_2 + x_1y_1 \\ i\sqrt{2}\bar{p}p_1p_2x_1y_3 - x_2y_1 \end{pmatrix}_{\mathbf{2},\mathbf{2'},\mathbf{2''}} \\ &\oplus \begin{pmatrix} -i\sqrt{2}pp_2x_2y_3 + x_1y_2 \\ i\sqrt{2}\bar{p}\bar{p}_1x_1y_1 - x_2y_2 \end{pmatrix}_{\mathbf{2'},\mathbf{2''},\mathbf{2}} \\ &\oplus \begin{pmatrix} -i\sqrt{2}p\bar{p}_1\bar{p}_2x_2y_1 + x_1y_3 \\ i\sqrt{2}\bar{p}\bar{p}_2x_1y_2 - x_2y_3 \end{pmatrix}_{\mathbf{2''},\mathbf{2},\mathbf{2'}}, \end{aligned} \tag{116}$$

$$(x)_{\mathbf{1'(1'')}} \otimes \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}_{\mathbf{2},\mathbf{2'},\mathbf{2''}} = \begin{pmatrix} xy_1 \\ xy_2 \end{pmatrix}_{\mathbf{2'(2''),2''(2),2(2')}}, \tag{117}$$

$$(x)_{\mathbf{1'}} \otimes \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}_{\mathbf{3}} = \begin{pmatrix} xy_3 \\ \bar{p}_1^2\bar{p}_2xy_1 \\ \bar{p}_1\bar{p}_2^2xy_2 \end{pmatrix}_{\mathbf{3}}, \quad (x)_{\mathbf{1''}} \otimes \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}_{\mathbf{3}} = \begin{pmatrix} xy_2 \\ \bar{p}_1p_2xy_3 \\ \bar{p}_1^2\bar{p}_2xy_1 \end{pmatrix}_{\mathbf{3}}. \tag{118}$$

	h	χ_1	$\chi_{1'}$	$\chi_{1''}$	χ_2	$\chi_{2'}$	$\chi_{2''}$	χ_3
C_1	1	1	1	1	2	2	2	3
$C_{1'}$	2	1	1	1	-2	-2	-2	3
C_4	3	1	ω	ω^2	-1	$-\omega$	$-\omega^2$	0
C_4'	3	1	ω^2	ω	-1	$-\omega^2$	$-\omega$	0
$C_{4''}$	6	1	ω	ω^2	1	ω	ω^2	0
$C_{4'''}$	6	1	ω^2	ω	1	ω^2	ω	0
C_6	4	1	1	1	0	0	0	-1

Table 6: Characters of T' representations

The representations for p' can be in general obtained by transforming p as follows,

$$\Phi_2(p') = \begin{pmatrix} 1 & 0 \\ 0 & e^{-i\gamma} \end{pmatrix} \Phi_2(p), \quad p' = pe^{i\gamma}, \tag{119}$$

$$\Phi_3(p') = \begin{pmatrix} 1 & 0 & 0 \\ 0 & e^{-i\gamma} & 0 \\ 0 & 0 & e^{-i(\alpha+\beta)} \end{pmatrix} \Phi_3(p), \quad p_1' = p_1 e^{i\alpha}, \quad p_2' = p_2 e^{-i\beta}. \tag{120}$$

If one takes the parameters $p = i$ and $p_1 = p_2 = 1$, then the generator s is simplified as

$$s = -\frac{i}{\sqrt{3}} \begin{pmatrix} 1 & \sqrt{2} \\ \sqrt{2} & -1 \end{pmatrix}, \quad \text{on } \mathbf{2}, \tag{121}$$

$$s = \begin{pmatrix} -1 & 2 & 2 \\ 2 & -1 & 2 \\ 2 & 2 & -1 \end{pmatrix}, \quad \text{on } \mathbf{3}.$$

These tensor products can be also simplified as

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}_{\mathbf{2}(\mathbf{2}')} \otimes \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}_{\mathbf{2}(\mathbf{2}'')} = \left(\frac{x_1y_2 - x_2y_1}{\sqrt{2}}\right)_{\mathbf{1}} \oplus \begin{pmatrix} \frac{x_1y_2+x_2y_1}{\sqrt{2}} \\ -x_1y_1 \\ x_2y_2 \end{pmatrix}_{\mathbf{3}}, \tag{122}$$

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}_{\mathbf{2}'(\mathbf{2})} \otimes \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}_{\mathbf{2}'(\mathbf{2}'')} = \left(\frac{x_1y_2 - x_2y_1}{\sqrt{2}}\right)_{\mathbf{1}''} \oplus \begin{pmatrix} -x_1y_1 \\ x_2y_2 \\ \frac{x_1y_2+x_2y_1}{\sqrt{2}} \end{pmatrix}_{\mathbf{3}}, \tag{123}$$

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}_{\mathbf{2}''(\mathbf{2})} \otimes \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}_{\mathbf{2}''(\mathbf{2}')} = \left(\frac{x_1y_2 - x_2y_1}{\sqrt{2}}\right)_{\mathbf{1}'} \oplus \begin{pmatrix} x_2y_2 \\ \frac{x_1y_2+x_2y_1}{\sqrt{2}} \\ -x_1y_1 \end{pmatrix}_{\mathbf{3}}, \tag{124}$$

$$
\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}_{\mathbf{3}} \otimes \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}_{\mathbf{3}} \begin{array}{ll} = & [x_1y_1 + x_2y_3 + x_3y_2]_{\mathbf{1}} \\ \oplus & [x_3y_3 + x_1y_2 + x_2y_1]_{\mathbf{1'}} \oplus (x_2y_2 + x_1y_3 + x_3y_1]_{\mathbf{1''}} \\ \oplus & \begin{pmatrix} 2x_1y_1 - x_2y_3 - x_3y_3 \\ 2x_3y_3 - x_1y_2 - x_2y_1 \\ 2x_2y_2 - x_1y_3 - x_3y_1 \end{pmatrix}_{\mathbf{3}} \\ \oplus & \begin{pmatrix} x_2y_3 - x_3y_2 \\ x_1y_2 - x_2y_1 \\ x_3y_1 - x_1y_3 \end{pmatrix}_{\mathbf{3}} , \end{array} \tag{125}
$$

$$
\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}_{\mathbf{2},\mathbf{2'},\mathbf{2''}} \otimes \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}_{\mathbf{3}} \begin{array}{ll} = & \begin{pmatrix} \sqrt{2}x_2y_2 + x_1y_1 \\ \sqrt{2}x_1y_3 - x_2y_1 \end{pmatrix}_{\mathbf{2},\mathbf{2'},\mathbf{2''}} \oplus \begin{pmatrix} \sqrt{2}x_2y_3 + x_1y_2 \\ \sqrt{2}x_1y_1 - x_2y_2 \end{pmatrix}_{\mathbf{2'},\mathbf{2''},\mathbf{2}} \\ \oplus & \begin{pmatrix} \sqrt{2}x_2y_1 + x_1y_3 \\ \sqrt{2}x_1y_2 - x_2y_3 \end{pmatrix}_{\mathbf{2''},\mathbf{2},\mathbf{2'}} , \end{array} \tag{126}
$$

$$
(x)_{\mathbf{1'}(\mathbf{1''})} \otimes \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}_{\mathbf{2},\mathbf{2'},\mathbf{2''}} = \begin{pmatrix} xy_1 \\ xy_2 \end{pmatrix}_{\mathbf{2'}(\mathbf{2''}),\mathbf{2''}(\mathbf{2}),\mathbf{2}(\mathbf{2'})} , \tag{127}
$$

$$
(x)_{\mathbf{1'}} \otimes \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}_{\mathbf{3}} = \begin{pmatrix} xy_3 \\ xy_1 \\ xy_2 \end{pmatrix}_{\mathbf{3}} , \quad (x)_{\mathbf{1''}} \otimes \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}_{\mathbf{3}} = \begin{pmatrix} xy_2 \\ xy_3 \\ xy_1 \end{pmatrix}_{\mathbf{3}} . \tag{128}
$$

When $p = e^{i\pi/12}$ and $p_1 = p_2 = \omega$, then the representation and their tensor products are given in the Ref. [158].

6 D_N

6.1 Generic aspects

D_N is a symmetry of a regular polygon with N sides and it is often called as the dihedral group. It is isomorphic to $Z_N \rtimes Z_2$ and it is also denoted by $\Delta(2N)$. It consists of cyclic rotation, Z_N and reflection. That is, it is generated by two generators a and b, which act on N edges x_i $(i = 1, \cdots, N)$ of N-polygon as

$$\begin{aligned} a &: (x_1, x_2 \cdots, x_N) \to (x_N, x_1 \cdots, x_{N-1}), \quad (129) \\ b &: (x_1, x_2 \cdots, x_N) \to (x_1, x_N \cdots, x_2). \quad (130) \end{aligned}$$

These two generators satisfy

$$a^N = e, \qquad b^2 = e, \qquad bab = a^{-1}, \quad (131)$$

where the third equation is equivalent to $aba = b$. The order of D_N is equal to $2N$, and all of $2N$ elements are written as $a^m b^k$ with $m = 0, \cdots, N-1$ and $k = 0, 1$. The third equation in (131) implies that the Z_N subgroup including a^m is a normal subgroup of D_N. Thus, D_N corresponds to a semi-direct product between Z_N including a^m and Z_2 including b^k, i.e. $Z_N \rtimes Z_2$. Eq. (129) corresponds to the (reducible) N-dimensional representation. The simple doublet representation is written as

$$a = \begin{pmatrix} \cos 2\pi/N & -\sin 2\pi/N \\ \sin 2\pi/N & \cos 2\pi/N \end{pmatrix}, \qquad b = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \quad (132)$$

• **Conjugacy classes**

Because of the algebraic relations (131), it is found that a^m and a^{N-m} belong to the same conjugacy class and also b and $a^{2m}b$ belong to the same conjugacy class. When N is even, D_N has the following $3 + N/2$ conjugacy classes,

$$\begin{array}{ccc} C_1 : & \{e\}, & h = 1, \\ C_2^{(1)} : & \{a, a^{N-1}\}, & h = N, \\ \vdots & \vdots & \vdots, \\ C_2^{(N/2-1)} : & \{a^{N/2-1}, a^{N/2+1}\}, & h = N/gcd(N, N/2-1), \\ C_1' : & \{a^{N/2}\}, & h = 2, \\ C_{N/2} : & \{b, a^2b, \cdots, a^{N-2}b\}, & h = 2, \\ C_{N/2}' : & \{ab, a^3b, \cdots, a^{N-1}b\}, & h = 2, \end{array} \quad (133)$$

where we have also shown the orders of each element in the conjugacy class by h. That implies that there are $3+N/2$ irreducible representations. Furthermore, the orthogonality relation (12) requires

$$\sum_\alpha [\chi_\alpha(C_1)]^2 = \sum_n m_n n^2 = m_1 + 4m_2 + 9m_3 + \cdots = 2N, \quad (134)$$

for m_i, which satisfies $m_1+m_2+m_3+\cdots = 3+N/2$. The solution is found as $(m_1, m_2) = (4, N/2-1)$. That is, there are four singlets and $(N/2-1)$ doublets.

On the other hand, when N is odd, D_N has the following $2+(N-1)/2$ conjugacy classes,

$$\begin{array}{ccc} C_1 : & \{e\}, & h=1, \\ C_2^{(1)} : & \{a, a^{N-1}\}, & h=N, \\ \vdots & \vdots & \vdots, \\ C_2^{(N-1)/2} : & \{a^{(N-1)/2}, a^{(N+1)/2}\}, & h=N/gcd(N,(N-1)/2), \\ C_N : & \{b, ab, \cdots, a^{N-1}b\}, & h=2. \end{array} \tag{135}$$

That is, there are $2+(N-1)/2$ irreducible representations. Furthermore, the orthogonality relation (12) requires the same equation as (134) for m_i, which satisfies $m_1+m_2+m_3+\cdots = 2+(N-1)/2$. The solution is found as $(m_1, m_2) = (2, (N-1)/2)$. That is, there are two singlets and $(N-1)/2$ doublets.

• **Characters and representations**

First of all, we study on singlets. When N is even, there are four singlets. Because of $b^2 = e$ in $C_{N/2}$ and $(ab)^2 = e$ in $C'_{N/2}$, the characters $\chi_\alpha(g)$ for four singlets should satisfy $\chi_\alpha(C_{N/2}) = \pm 1$ and $\chi_\alpha(C'_{N/2}) = \pm 1$. Thus, we have four possible combinations of $\chi_\alpha(C_{N/2}) = \pm 1$ and $\chi_\alpha(C'_{N/2}) = \pm 1$ and they correspond to four singlets, $\mathbf{1}_{\pm\pm}$, which are shown in Table 7.

Similarly, we can study D_N with $N =$ odd, which has two singlets. Because of $b^2 = e$ in C_N, the characters $\chi_\alpha(g)$ for two singlets should satisfy $\chi_\alpha(C_N) = \pm 1$. Since both b and ab belong to the same conjugacy class C_N, the characters $\chi_\alpha(a)$ for two singlets must always satisfy $\chi_\alpha(a) = 1$. Thus, there are two singlets, $\mathbf{1}_+$ and $\mathbf{1}_-$. Their characters are determined by whether the conjugacy class includes b or not as shown in Table 8.

Next, we study doublet representations, that is, (2×2) matrix representations. Indeed, Eq. (132) corresponds to one of doublet representations. Similarly, (2×2) matrix representations for generic doublet $\mathbf{2}_k$ are obtained by replacing

$$a \to a^k. \tag{136}$$

That is, a and b are represented for the doublet $\mathbf{2}_k$ as

$$a = \begin{pmatrix} \cos 2\pi k/N & -\sin 2\pi k/N \\ \sin 2\pi k/N & \cos 2\pi k/N \end{pmatrix}, \qquad b = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \tag{137}$$

where $k = 1, \cdots, N/2-1$ for $N =$ even and $k = 1, \cdots, (N-1)/2$ for $N =$ odd. When we write the doublet $\mathbf{2}_k$ as

$$\mathbf{2}_k = \begin{pmatrix} x_k \\ y_k \end{pmatrix}, \tag{138}$$

the generator a is the Z_N rotation on the two-dimensional real coordinates (x_k, y_k) and the generator b is the reflection along y_k, i.e. $y_k \to -y_k$. These transformations can

	h	$\chi_{1_{++}}$	$\chi_{1_{+-}}$	$\chi_{1_{-+}}$	$\chi_{1_{--}}$	χ_{2k}
C_1	1	1	1	1	1	2
C_2^1	N	1	-1	-1	1	$2\cos(2\pi k/N)$
$\vdots$						
$C_2^{N/2-1}$	$N/gcd(N, N/2-1)$	1	$(-1)^{(N/2-1)}$	$(-1)^{(N/2-1)}$	1	$2\cos(2\pi k(N/2-1)/N)$
C_1'	2	1	$(-1)^{N/2}$	$(-1)^{N/2}$	1	-2
$C_{N/2}$	2	1	1	-1	-1	0
$C'_{N/2}$	2	1	-1	1	-1	0

Table 7: Characters of $D_{N=\text{even}}$ representations

	h	χ_{1_+}	χ_{1_-}	χ_{2k}
C_1	1	1	1	2
C_2^1	N	1	1	$2\cos(2\pi k/N)$
$\vdots$				
$C_2^{(N-1)/2}$	$N/gcd(N, (N-1)/2)$	1	1	$2\cos(2\pi k(N-1)/2N)$
C_N	2	1	-1	0

Table 8: Characters of $D_{N=\text{odd}}$ representations

be represented on the complex coordinate z_k and its conjugate $\bar{z}_{-k}$. These bases are transformed as

$$\begin{pmatrix} z_k \\ \bar{z}_{-k} \end{pmatrix} = U \begin{pmatrix} x_k \\ y_k \end{pmatrix}, \qquad U = \begin{pmatrix} 1 & i \\ 1 & -i \end{pmatrix}. \tag{139}$$

Then, in the complex basis, the generators, a and b, can be obtained as $\tilde{a} = UaU^{-1}$ and $\tilde{b} = UbU^{-1}$,

$$\tilde{a} = \begin{pmatrix} \exp 2\pi ik/N & 0 \\ 0 & \exp -2\pi ik/N \end{pmatrix}, \qquad \tilde{b} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \tag{140}$$

This complex basis may be useful. For instance, the generator $\tilde{a}$ is the diagonal matrix. That implies that in the doublet $\mathbf{2}_k$, which is denoted by

$$\mathbf{2}_k = \begin{pmatrix} z_k \\ \bar{z}_{-k} \end{pmatrix}, \tag{141}$$

each of up and down components, z_k and $\bar{z}_{-k}$, has the definite Z_N charge. That is, Z_N charges of z_k and $\bar{z}_{-k}$ are equal to k and $-k$, respectively. The characters of these matrices for the doublets $\mathbf{2}_k$ are obtained and those are shown in Tables 7 and 8. These characters satisfy the orthogonality relations (10) and (11).

• **Tensor products**

Now, we study the tensor products. First, we consider the D_N group with $N =$ even. Let us start with $\mathbf{2}_k \times \mathbf{2}_{k'}$, i.e.

$$\begin{pmatrix} z_k \\ \bar{z}_{-k} \end{pmatrix}_{\mathbf{2}_k} \otimes \begin{pmatrix} z_{k'} \\ \bar{z}_{-k'} \end{pmatrix}_{\mathbf{2}_{k'}}, \tag{142}$$

where $k, k' = 1, \cdots, N/2-1$. Note that $z_k z_{k'}$, $z_k \bar{z}_{-k'}$, $\bar{z}_{-k} z_{k'}$ and $\bar{z}_{-k}\bar{z}_{-k'}$ have define Z_N changes, i.e. $k+k'$, $k-k'$, $-k+k'$ and $-k-k'$, respectively. For the case with $k+k' \neq N/2$ and $k-k' \neq 0$, they decompose two doublets as

$$\begin{pmatrix} z_k \\ \bar{z}_{-k} \end{pmatrix}_{\mathbf{2}_k} \otimes \begin{pmatrix} z_{k'} \\ \bar{z}_{-k'} \end{pmatrix}_{\mathbf{2}_{k'}} = \begin{pmatrix} z_k z_{k'} \\ \bar{z}_{-k}\bar{z}_{-k'} \end{pmatrix}_{\mathbf{2}_{k+k'}} \oplus \begin{pmatrix} z_k \bar{z}_{-k'} \\ \bar{z}_{-k} z_{k'} \end{pmatrix}_{\mathbf{2}_{k-k'}}. \tag{143}$$

When $k + k' = N/2$, the matrix a is represented on the above (reducible) doublet $(z_k z_{k'}, \bar{z}_{-k}\bar{z}_{-k'})$ as

$$a \begin{pmatrix} z_k z_{k'} \\ \bar{z}_{-k}\bar{z}_{-k'} \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} z_k z_{k'} \\ \bar{z}_{-k}\bar{z}_{-k'} \end{pmatrix}. \tag{144}$$

Since a is proportional to the (2×2) identity matrix for $(z_k z_{k'}, \bar{z}_{-k}\bar{z}_{-k'})$ with $k+k' = N/2$, we can diagonalize another matrix b in this vector space $(z_k z_{k'}, \bar{z}_{-k}\bar{z}_{-k'})$. Such a basis is obtained as $(z_k z_{k'} + \bar{z}_{-k}\bar{z}_{-k'}, z_k z_{k'} - \bar{z}_{-k}\bar{z}_{-k'})$ and their eigenvalues of b are obtained as

$$b \begin{pmatrix} z_k z_{k'} + \bar{z}_{-k}\bar{z}_{-k'} \\ z_k z_{k'} - \bar{z}_{-k}\bar{z}_{-k'} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} z_k z_{k'} + \bar{z}_{-k}\bar{z}_{-k'} \\ z_k z_{k'} - \bar{z}_{-k}\bar{z}_{-k'} \end{pmatrix}. \tag{145}$$

Thus, $z_k z_{k'} + \bar{z}_{-k}\bar{z}_{-k'}$ and $z_k z_{k'} - \bar{z}_{-k}\bar{z}_{-k'}$ correspond to $\mathbf{1}_{+-}$ and $\mathbf{1}_{-+}$, respectively.

In the case of $k - k' = 0$, a similar decomposition happens for the (reducible) doublet $(z_k \bar{z}_{-k'}, \bar{z}_{-k} z_{k'})$. The matrix a is the (2×2) identity matrix on the vector space $(z_k \bar{z}_{-k'}, \bar{z}_{-k} z_{k'})$ with $k - k' = 0$. Then, we take the basis $(z_k \bar{z}_{-k'} + \bar{z}_{-k} z_{k'}, z_k \bar{z}_{-k'} - \bar{z}_{-k} z_{k'})$, where b is diagonalized. That is, $z_k \bar{z}_{-k'} + \bar{z}_{-k} z_{k'}$ and $z_k \bar{z}_{-k'} - \bar{z}_{-k} z_{k'}$ correspond to $\mathbf{1}_{++}$ and $\mathbf{1}_{--}$, respectively.

Next, we study the tensor products of the doublets $\mathbf{2}_k$ and singlets, e.g. $\mathbf{1}_{--} \times \mathbf{2}_k$. Here we denote the vector space for the singlet $\mathbf{1}_{--}$ by w, where $aw = w$ and $bw = -w$. Then, it is found that $(wz_k, -w\bar{z}_k)$ is nothing but the doublet $\mathbf{2}_k$, that is, $\mathbf{1}_{--} \times \mathbf{2}_k = \mathbf{2}_k$. Similar results are obtained for other singlets. Furthermore, it is straightforward to study the tensor products among singlets. Hence, the tensor products of D_N irreducible representations with N =even are summarized as

$$\begin{pmatrix} z_k \\ \bar{z}_{-k} \end{pmatrix}_{\mathbf{2}_k} \otimes \begin{pmatrix} z_{k'} \\ \bar{z}_{-k'} \end{pmatrix}_{\mathbf{2}_{k'}} = \begin{pmatrix} z_k z_{k'} \\ \bar{z}_{-k}\bar{z}_{-k'} \end{pmatrix}_{\mathbf{2}_{k+k'}} \oplus \begin{pmatrix} z_k \bar{z}_{-k'} \\ \bar{z}_{-k} z_{k'} \end{pmatrix}_{\mathbf{2}_{k-k'}}, \tag{146}$$

for $k + k' \neq N/2$ and $k - k' \neq 0$,

$$\begin{aligned} \begin{pmatrix} z_k \\ \bar{z}_{-k} \end{pmatrix}_{\mathbf{2}_k} \otimes \begin{pmatrix} z_{k'} \\ \bar{z}_{-k'} \end{pmatrix}_{\mathbf{2}_{k'}} &= (z_k z_{k'} + \bar{z}_{-k}\bar{z}_{-k'})_{\mathbf{1}_{+-}} \oplus (z_k z_{k'} - \bar{z}_{-k}\bar{z}_{-k'})_{\mathbf{1}_{-+}} \\ &\quad \oplus \begin{pmatrix} z_k \bar{z}_{-k'} \\ \bar{z}_{-k} z_{k'} \end{pmatrix}_{\mathbf{2}_{k-k'}}, \end{aligned} \tag{147}$$

for $k+k'=N/2$ and $k-k'\neq 0$,

$$\begin{pmatrix} z_k \\ \bar{z}_{-k} \end{pmatrix}_{\mathbf{2}_k} \otimes \begin{pmatrix} z_{k'} \\ \bar{z}_{-k'} \end{pmatrix}_{\mathbf{2}_{k'}} = (z_k\bar{z}_{-k'}+\bar{z}_{-k}z_{k'})_{\mathbf{1}_{++}} \oplus (z_k\bar{z}_{-k'}-\bar{z}_{-k}z_{k'})_{\mathbf{1}_{--}} \oplus \begin{pmatrix} z_k z_{k'} \\ \bar{z}_{-k}\bar{z}_{-k'} \end{pmatrix}_{\mathbf{2}_{k+k'}}, \tag{148}$$

for $k+k'\neq N/2$ and $k-k'=0$,

$$\begin{pmatrix} z_k \\ \bar{z}_{-k} \end{pmatrix}_{\mathbf{2}_k} \otimes \begin{pmatrix} z_{k'} \\ \bar{z}_{-k'} \end{pmatrix}_{\mathbf{2}_{k'}} = (z_k\bar{z}_{-k'}+\bar{z}_{-k}z_{k'})_{\mathbf{1}_{++}} \oplus (z_k\bar{z}_{-k'}-\bar{z}_{-k}z_{k'})_{\mathbf{1}_{--}} \oplus (z_k z_{k'}+\bar{z}_{-k}\bar{z}_{-k'})_{\mathbf{1}_{+-}} \oplus (z_k z_{k'}-\bar{z}_{-k}\bar{z}_{-k'})_{\mathbf{1}_{-+}}, \tag{149}$$

for $k+k'=N/2$ and $k-k'=0$,

$$(w)_{\mathbf{1}_{++}} \otimes \begin{pmatrix} z_k \\ \bar{z}_{-k} \end{pmatrix}_{\mathbf{2}_k} = \begin{pmatrix} wz_k \\ w\bar{z}_{-k} \end{pmatrix}_{\mathbf{2}_k}, \quad (w)_{\mathbf{1}_{--}} \otimes \begin{pmatrix} z_k \\ \bar{z}_{-k} \end{pmatrix}_{\mathbf{2}_k} = \begin{pmatrix} wz_k \\ -w\bar{z}_{-k} \end{pmatrix}_{\mathbf{2}_k},$$
$$(w)_{\mathbf{1}_{+-}} \otimes \begin{pmatrix} z_k \\ \bar{z}_{-k} \end{pmatrix}_{\mathbf{2}_k} = \begin{pmatrix} w\bar{z}_{-k} \\ wz_k \end{pmatrix}_{\mathbf{2}_k}, \quad (w)_{\mathbf{1}_{-+}} \otimes \begin{pmatrix} z_k \\ \bar{z}_{-k} \end{pmatrix}_{\mathbf{2}_k} = \begin{pmatrix} w\bar{z}_{-k} \\ -wz_k \end{pmatrix}_{\mathbf{2}_k}, \tag{150}$$

$$\mathbf{1}_{s_1s_2} \otimes \mathbf{1}_{s_1's_2'} = \mathbf{1}_{s_1''s_2''}, \tag{151}$$

where $s_1''=s_1s_1'$ and $s_2''=s_2s_2'$.

Similarly, we can analyze the tensor products of D_N irreducible representations with $N=$ odd. Its results are summarized as follows,

$$\begin{pmatrix} z_k \\ \bar{z}_{-k} \end{pmatrix}_{\mathbf{2}_k} \otimes \begin{pmatrix} z_{k'} \\ \bar{z}_{-k'} \end{pmatrix}_{\mathbf{2}_{k'}} = \begin{pmatrix} z_k z_{k'} \\ \bar{z}_{-k}\bar{z}_{-k'} \end{pmatrix}_{\mathbf{2}_{k+k'}} \oplus \begin{pmatrix} z_k\bar{z}_{-k'} \\ \bar{z}_{-k}z_{k'} \end{pmatrix}_{\mathbf{2}_{k-k'}}, \tag{152}$$

for $k-k'\neq 0$, where $k,k'=1,\cdots,N/2-1$,

$$\begin{pmatrix} z_k \\ \bar{z}_{-k} \end{pmatrix}_{\mathbf{2}_k} \otimes \begin{pmatrix} z_{k'} \\ \bar{z}_{-k'} \end{pmatrix}_{\mathbf{2}_{k'}} = (z_k\bar{z}_{-k'}+\bar{z}_{-k}z_{k'})_{\mathbf{1}_{+}} \oplus (z_k\bar{z}_{-k'}-\bar{z}_{-k}z_{k'})_{\mathbf{1}_{-}} \oplus \begin{pmatrix} z_k z_{k'} \\ \bar{z}_{-k}\bar{z}_{-k'} \end{pmatrix}_{\mathbf{2}_{k+k'}}, \tag{153}$$

for $k-k'=0$,

$$(w)_{\mathbf{1}_{++}} \otimes \begin{pmatrix} z_k \\ \bar{z}_{-k} \end{pmatrix}_{\mathbf{2}_k} = \begin{pmatrix} wz_k \\ w\bar{z}_{-k} \end{pmatrix}_{\mathbf{2}_k}, \qquad (w)_{\mathbf{1}_{-}} \otimes \begin{pmatrix} z_k \\ \bar{z}_{-k} \end{pmatrix}_{\mathbf{2}_k} = \begin{pmatrix} wz_k \\ -w\bar{z}_{-k} \end{pmatrix}_{\mathbf{2}_k}, \tag{154}$$

$$\mathbf{1}_{s} \otimes \mathbf{1}_{s'} = \mathbf{1}_{s''}, \tag{155}$$

where $s''=ss'$.

Note that the above multiplication rules are the same between the complex basis and the real basis. For example, we obtain that in both bases $\mathbf{2}_k \otimes \mathbf{2}_{k'} = \mathbf{2}'_{k+k} + \mathbf{2}_{k-k'}$ for $k+k'\neq N/2$ and $k-k'\neq 0$. However, elements of doublets are written in a different way, although those transform as (139).

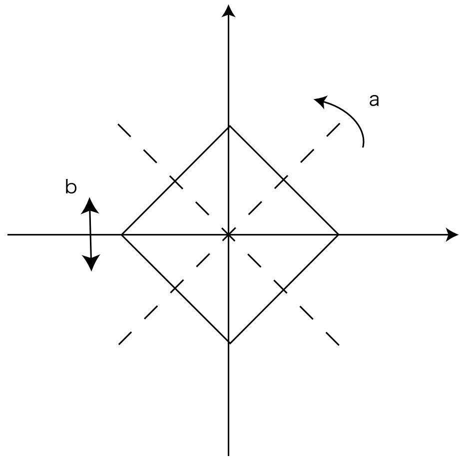

Figure 5: The D_4 symmetry of a square

6.2 D_4

Here, we give simple examples of D_N. The smallest non-Abelian group in D_N is D_3. However, D_3 corresponds to a group of all possible permutations of three objects, that is, S_3. Thus, we show D_4 and D_5 as simple examples.

The D_4 is the symmetry of a square, which is generated by the $\pi/2$ rotation a and the reflection b, where they satisfy $a^4 = e$, $b^2 = e$ and $bab = a^{-1}$. (See Figure 5.) Indeed, the D_4 consists of the eight elements, $a^m b^k$ with $m = 0, 1, 2, 3$ and $k = 0, 1$. The D_4 has the following five conjugacy classes,

$$\begin{array}{lcl} C_1: & \{e\}, & h = 1, \\ C_2: & \{a, a^3\}, & h = 4, \\ C_1': & \{a^2\}, & h = 2, \\ C_2': & \{b, a^2 b\}, & h = 2, \\ C_2'': & \{ab, a^3 b\}, & h = 2, \end{array} \tag{156}$$

where we have also shown the orders of each element in the conjugacy class by h.

The D_4 has four singlets, $\mathbf{1}_{++}$, $\mathbf{1}_{+-}$, $\mathbf{1}_{-+}$ and $\mathbf{1}_{--}$, and one doublet $\mathbf{2}$. The characters are shown in Table 9. The tensor products are obtained as

$$\begin{aligned} \begin{pmatrix} z \\ \bar{z} \end{pmatrix}_{\mathbf{2}} \otimes \begin{pmatrix} z' \\ \bar{z}' \end{pmatrix}_{\mathbf{2}} &= (z\bar{z}' + \bar{z}z')_{\mathbf{1}_{++}} \oplus (z\bar{z}' - \bar{z}z')_{\mathbf{1}_{--}} \\ &\quad \oplus (zz' + \bar{z}\bar{z}')_{\mathbf{1}_{+-}} \oplus (zz' - \bar{z}\bar{z}')_{\mathbf{1}_{-+}}, \end{aligned} \tag{157}$$

$$\begin{aligned} &(w)_{\mathbf{1}_{++}} \otimes \begin{pmatrix} z \\ \bar{z} \end{pmatrix}_{\mathbf{2}} = \begin{pmatrix} wz \\ w\bar{z} \end{pmatrix}_{\mathbf{2}}, \qquad (w)_{\mathbf{1}_{--}} \otimes \begin{pmatrix} z \\ \bar{z} \end{pmatrix}_{\mathbf{2}} = \begin{pmatrix} wz \\ -w\bar{z} \end{pmatrix}_{\mathbf{2}}, \\ &(w)_{\mathbf{1}_{+-}} \otimes \begin{pmatrix} z \\ \bar{z} \end{pmatrix}_{\mathbf{2}} = \begin{pmatrix} w\bar{z} \\ wz \end{pmatrix}_{\mathbf{2}}, \qquad (w)_{\mathbf{1}_{-+}} \otimes \begin{pmatrix} z \\ \bar{z} \end{pmatrix}_{\mathbf{2}} = \begin{pmatrix} w\bar{z} \\ -wz \end{pmatrix}_{\mathbf{2}}, \end{aligned} \tag{158}$$

$$\mathbf{1}_{s_1 s_2} \otimes \mathbf{1}_{s_1' s_2'} = \mathbf{1}_{s_1'' s_2''}, \tag{159}$$

where $s_1'' = s_1 s_1'$ and $s_2'' = s_2 s_2'$.

	h	$\chi_{1_{++}}$	$\chi_{1_{+-}}$	$\chi_{1_{-+}}$	$\chi_{1_{--}}$	χ_2
C_1	1	1	1	1	1	2
C_2	4	1	-1	-1	1	0
C_1'	2	1	1	1	1	-2
C_2'	2	1	1	-1	-1	0
C_2''	2	1	-1	1	-1	0

Table 9: Characters of D_4 representations

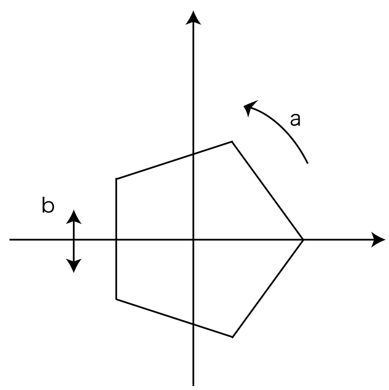

Figure 6: The D_5 symmetry of a regular pentagon

6.3 D_5

The D_5 is the symmetry of a regular pentagon, which is generated by the $2\pi/5$ rotation a and the reflection b. See Figure 6. They satisfy that $a^5 = e$, $b^2 = e$ and $bab = a^{-1}$. The D_5 includes the 10 elements, $a^m b^k$ with $m = 0, 1, 2, 3, 4$ and $k = 0, 1$. They are classified into the following four conjugacy classes,

$$\begin{array}{rcl} C_1 : & \{e\}, & h = 1, \\ C_2^{(1)} : & \{a, a^4\}, & h = 5, \\ C_2^{(2)} : & \{a^2, a^3\}, & h = 5, \\ C_5 : & \{b, ab, a^2b, a^3b, a^4b\}, & h = 2. \end{array} \tag{160}$$

The D_5 has two singlets, $\mathbf{1}_+$ and $\mathbf{1}_-$, and two doublets, $\mathbf{2}_1$ and $\mathbf{2}_2$. Their characters are shown in Table 10.

The tensor products are obtained as

$$\begin{pmatrix} z \\ \bar{z} \end{pmatrix}_{\mathbf{2}_2} \otimes \begin{pmatrix} z' \\ \bar{z}' \end{pmatrix}_{\mathbf{2}_1} = \begin{pmatrix} zz' \\ \bar{z}\bar{z}' \end{pmatrix}_{\mathbf{2}_2} \oplus \begin{pmatrix} z\bar{z}' \\ \bar{z}z' \end{pmatrix}_{\mathbf{2}_1}, \tag{161}$$

$$\begin{pmatrix} z_k \\ \bar{z}_{-k} \end{pmatrix}_{\mathbf{2}_k} \otimes \begin{pmatrix} z'_k \\ \bar{z}'_{-k} \end{pmatrix}_{\mathbf{2}_k} = \left(z_k\bar{z}'_{-k} + \bar{z}_{-k}z'_k\right)_{\mathbf{1}_+} \oplus \left(z_k\bar{z}'_{-k} - \bar{z}_{-k}z'_k\right)_{\mathbf{1}_-} \oplus \begin{pmatrix} z_k z'_k \\ \bar{z}_{-k}\bar{z}'_{-k} \end{pmatrix}_{\mathbf{2}_{2k}}, \tag{162}$$

	h	χ_{1_+}	χ_{1_-}	χ_{2_1}	χ_{2_2}
C_1	1	1	1	2	2
C_2^1	5	1	1	$2\cos(2\pi/5)$	$2\cos(4\pi/5)$
C_2^2	5	1	1	$2\cos(4\pi/5)$	$2\cos(8\pi/5)$
C_5	2	1	-1	0	0

Table 10: Characters of D_5 representations

$$(w)_{\mathbf{1}_+} \otimes \begin{pmatrix} z_k \\ \bar{z}_{-k} \end{pmatrix}_{\mathbf{2}_k} = \begin{pmatrix} wz_k \\ w\bar{z}_{-k} \end{pmatrix}_{\mathbf{2}_k}, \qquad (w)_{\mathbf{1}_-} \otimes \begin{pmatrix} z_k \\ \bar{z}_{-k} \end{pmatrix}_{\mathbf{2}_k} = \begin{pmatrix} wz_k \\ -w\bar{z}_{-k} \end{pmatrix}_{\mathbf{2}_k}, \quad (163)$$

$$\mathbf{1}_s \otimes \mathbf{1}_{s'} = \mathbf{1}_{s''}, \quad (164)$$

where $s'' = ss'$.

7 Q_N

7.1 Generic aspects

The binary dihedral group Q_N with $N =$ even consists of the elements, $a^m b^k$ with $m = 0, \cdots, N-1$ and $k = 0, 1$, where the generators a and b satisfy

$$a^N = e, \qquad b^2 = (a^{N/2}), \qquad b^{-1}ab = a^{-1}. \tag{165}$$

The order of Q_N is equal to $2N$. The generator a can be represented by the same (2×2) matrices as D_N, i.e.

$$a = \begin{pmatrix} \exp 2\pi ik/N & 0 \\ 0 & \exp -2\pi ik/N \end{pmatrix}. \tag{166}$$

Note that $a^{N/2} = e$ for $k =$ even and $a^{N/2} = -e$ for $k =$ odd. That leads to that $b^2 = e$ for $k =$ even and $b^2 = -e$ for $k =$ odd. Thus, the generators a and b are represented by (2×2) matrices, e.g. as

$$a = \begin{pmatrix} \exp 2\pi ik/N & 0 \\ 0 & \exp -2\pi ik/N \end{pmatrix}, \qquad b = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, \tag{167}$$

for $k =$ odd,

$$a = \begin{pmatrix} \exp 2\pi ik/N & 0 \\ 0 & \exp -2\pi ik/N \end{pmatrix}, \qquad b = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \tag{168}$$

for $k =$ even.

• **Conjugacy classes**

By use of the algebraic relations (165), the elements are classified into the $(3 + N/2)$ conjugacy classes as

$$\begin{array}{ccc} C_1 : & \{e\}, & h = 1, \\ C_2^{(1)} : & \{a, a^{N-1}\}, & h = N, \\ \vdots & \vdots & \vdots, \\ C_2^{(N/2-1)} : & \{a^{N/2-1}, a^{N/2+1}\}, & h = N/gcd(N, N/2-1), \\ C_1' : & \{a^{N/2}\}, & h = 2, \\ C_{N/2} : & \{b, a^2b, \cdots, a^{N-2}B\}, & h = 4, \\ C_{N/2}' : & \{ab, a^3b, \cdots, a^{N-1}B\}, & h = 4, \end{array} \tag{169}$$

where we have also shown the orders of each element in the conjugacy class by h. These are almost the same as the conjugacy classes of D_N with $N =$ even. There must be the $(3 + N/2)$ irreducible representations, and similarly to $D_{N=\text{ even}}$ there are four singlets and $(N/2 - 1)$ doublets.

	h	$\chi_{1_{++}}$	$\chi_{1_{+-}}$	$\chi_{1_{-+}}$	$\chi_{1_{--}}$	χ_{2_k}
C_1	1	1	1	1	1	2
C_2^1	N	1	-1	-1	1	$2\cos(2\pi k/N)$
$\vdots$						
$C_2^{N/2-1}$	$N/gcd(N,N/2-1)$	1	$(-1)^{(N/2-1)}$	$(-1)^{(N/2-1)}$	1	$2\cos(2\pi k(N/2-1)/N)$
C_1'	2	1	$(-1)^{N/2}$	$(-1)^{N/2}$	1	-2
$C_{N/2}$	4	1	1	-1	-1	0
$C_{N/2}'$	4	1	-1	1	-1	0

Table 11: Characters of Q_N representations for $N = 4n$

• Characters and representations

The characters of Q_N for doublets are the same as those of $D_{N=\text{ even}}$, and are shown in Tables 11 and 12. The characters of Q_N for singlets depend on the value of N. First, we consider the case with $N = 4n$, where we have the relation,

$$b^2 = a^{2n}. \tag{170}$$

Because of $b^4 = e$ in $C_{N/2}$, the characters $\chi_\alpha(b)$ for four singlets must satisfy $\chi_\alpha(b) = e^{\pi in/2}$ with $n = 0, 1, 2, 3$. In addition, the element ba^2 belongs to the same conjugacy class as b. That implies $\chi_\alpha(a^2) = 1$ for four singlets. Then, by using Eq. (170), we find $\chi_\alpha(b^2) = 1$, that is, $\chi_\alpha(b) = \pm 1$. Thus, the characters of Q_N with $N = 4n$ for singlets are the same as those of $D_{N=\text{ even}}$ and are shown in Table 11.

Next, we consider four singlets of Q_N for $N = 4n + 2$ and in this case we have the relation,

$$b^2 = a^{2n+1}. \tag{171}$$

Since b and a^2b are included in the same conjugacy class, the characters $\chi_\alpha(a^2)$ for four singlets must satisfy $\chi_\alpha(a^2) = 1$, that is, there are two possibilities, $\chi_\alpha(a) = \pm 1$. When $\chi_\alpha(a) = 1$, the relation (171) leads to the two possibilities $\chi_\alpha(b) = \pm 1$. On the other hand, when $\chi_\alpha(a) = -1$, the relation (171) leads to the two possibilities $\chi_\alpha(b) = \pm i$. Then, totally there are four possibilities corresponding to the four singlets. Note that $\chi_\alpha(a) = \chi_\alpha(b^2)$ for all of singlets.

• Tensor products

Furthermore, similarly to D_N with $N =$ even, the tensor products of Q_N irreducible representations can be analyzed. The results for Q_N with $N = 4n$ are obtained as

$$\begin{pmatrix} z_k \\ \bar{z}_{-k} \end{pmatrix}_{\mathbf{2}_k} \otimes \begin{pmatrix} z_{k'} \\ \bar{z}_{-k'} \end{pmatrix}_{\mathbf{2}_{k'}} = \begin{pmatrix} z_k z_{k'} \\ (-1)^{kk'} \bar{z}_{-k}\bar{z}_{-k'} \end{pmatrix}_{\mathbf{2}_{k+k'}} \oplus \begin{pmatrix} z_k \bar{z}_{-k'} \\ (-1)^{kk'} \bar{z}_{-k} z_{k'} \end{pmatrix}_{\mathbf{2}_{k-k'}}, \tag{172}$$

	h	$\chi_{1_{++}}$	$\chi_{1_{+-}}$	$\chi_{1_{-+}}$	$\chi_{1_{--}}$	χ_{2k}
C_1	1	1	1	1	1	2
C_2^1	N	1	-1	-1	1	$2\cos(2\pi k/N)$
$\vdots$						
$C_2^{N/2-1}$	$N/gcd(N,N/2-1)$	1	$(-1)^{(N/2-1)}$	$(-1)^{(N/2-1)}$	1	$2\cos(2\pi k(N/2-1)/N)$
C_1'	2	1	$(-1)^{N/2}$	$(-1)^{N/2}$	1	-2
$C_{N/2}$	4	1	i	$-i$	-1	0
$C'_{N/2}$	4	1	$-i$	i	-1	0

Table 12: Characters of Q_N representations for $N = 4n+2$

for $k+k' \neq N/2$ and $k-k' \neq 0$,

$$\begin{pmatrix} z_k \\ \bar{z}_{-k} \end{pmatrix}_{\mathbf{2}_k} \otimes \begin{pmatrix} z_{k'} \\ \bar{z}_{-k'} \end{pmatrix}_{\mathbf{2}_{k'}} = \left(z_k z_{k'} + (-1)^{kk'} \bar{z}_{-k}\bar{z}_{-k'}\right)_{\mathbf{1}_{+-}} \oplus \left(z_k z_{k'} - (-1)^{kk'} \bar{z}_{-k}\bar{z}_{-k'}\right)_{\mathbf{1}_{-+}} \oplus \begin{pmatrix} z_k \bar{z}_{-k'} \\ (-1)^{kk'} \bar{z}_{-k} z_{k'} \end{pmatrix}_{\mathbf{2}_{k-k'}}, \tag{173}$$

for $k+k' = N/2$ and $k-k' \neq 0$,

$$\begin{pmatrix} z_k \\ \bar{z}_{-k} \end{pmatrix}_{\mathbf{2}_k} \otimes \begin{pmatrix} z_{k'} \\ \bar{z}_{-k'} \end{pmatrix}_{\mathbf{2}_{k'}} = \left(z_k \bar{z}_{-k'} + (-1)^{kk'} \bar{z}_{-k} z_{k'}\right)_{\mathbf{1}_{++}} \oplus \left(z_k \bar{z}_{-k'} - (-1)^{kk'} \bar{z}_{-k} z_{k'}\right)_{\mathbf{1}_{--}} \oplus \begin{pmatrix} z_k z_{k'} \\ (-1)^{kk'} \bar{z}_{-k}\bar{z}_{-k'} \end{pmatrix}_{\mathbf{2}_{k+k'}}, \tag{174}$$

for $k+k' \neq N/2$ and $k-k' = 0$,

$$\begin{pmatrix} z_k \\ \bar{z}_{-k} \end{pmatrix}_{\mathbf{2}_k} \otimes \begin{pmatrix} z_{k'} \\ \bar{z}_{-k'} \end{pmatrix}_{\mathbf{2}_{k'}} = \left(z_k \bar{z}_{-k'} + (-1)^{kk'} \bar{z}_{-k} z_{k'}\right)_{\mathbf{1}_{++}} \oplus \left(z_k \bar{z}_{-k'} - (-1)^{kk'} \bar{z}_{-k} z_{k'}\right)_{\mathbf{1}_{--}} \oplus \left(z_k z_{k'} + (-1)^{kk'} \bar{z}_{-k}\bar{z}_{-k'}\right)_{\mathbf{1}_{+-}} \oplus \left(z_k z_{k'} - (-1)^{kk'} \bar{z}_{-k}\bar{z}_{-k'}\right)_{\mathbf{1}_{-+}}, \tag{175}$$

for $k+k' = N/2$ and $k-k' = 0$,

$$(w)_{\mathbf{1}_{++}} \otimes \begin{pmatrix} z_k \\ \bar{z}_{-k} \end{pmatrix}_{\mathbf{2}_k} = \begin{pmatrix} w z_k \\ w\bar{z}_{-k} \end{pmatrix}_{\mathbf{2}_k}, \quad (w)_{\mathbf{1}_{--}} \otimes \begin{pmatrix} z_k \\ \bar{z}_{-k} \end{pmatrix}_{\mathbf{2}_k} = \begin{pmatrix} w z_k \\ -w\bar{z}_{-k} \end{pmatrix}_{\mathbf{2}_k},$$
$$(w)_{\mathbf{1}_{+-}} \otimes \begin{pmatrix} z_k \\ \bar{z}_{-k} \end{pmatrix}_{\mathbf{2}_k} = \begin{pmatrix} w\bar{z}_{-k} \\ w z_k \end{pmatrix}_{\mathbf{2}_k}, \quad (w)_{\mathbf{1}_{-+}} \otimes \begin{pmatrix} z_k \\ \bar{z}_{-k} \end{pmatrix}_{\mathbf{2}_k} = \begin{pmatrix} w\bar{z}_{-k} \\ -w z_k \end{pmatrix}_{\mathbf{2}_k}, \tag{176}$$

$$\mathbf{1}_{s_1 s_2} \otimes \mathbf{1}_{s_1' s_2'} = \mathbf{1}_{s_1'' s_2''}, \tag{177}$$

where $s_1'' = s_1 s_1'$ and $s_2'' = s_2 s_2'$. Note that some minus signs are different from the tensor products of D_N.

Similarly, the tensor products of Q_N with $N = 4n+2$ are obtained as

$$\begin{pmatrix} z_k \\ \bar{z}_{-k} \end{pmatrix}_{\mathbf{2}_k} \otimes \begin{pmatrix} z_{k'} \\ \bar{z}_{-k'} \end{pmatrix}_{\mathbf{2}_{k'}} = \begin{pmatrix} z_k z_{k'} \\ (-1)^{kk'} \bar{z}_{-k} \bar{z}_{-k'} \end{pmatrix}_{\mathbf{2}_{k+k'}} \oplus \begin{pmatrix} z_k \bar{z}_{-k'} \\ (-1)^{kk'} \bar{z}_{-k} z_{k'} \end{pmatrix}_{\mathbf{2}_{k-k'}}, \tag{178}$$

for $k + k' \neq N/2$ and $k - k' \neq 0$,

$$\begin{aligned} \begin{pmatrix} z_k \\ \bar{z}_{-k} \end{pmatrix}_{\mathbf{2}_k} \otimes \begin{pmatrix} z_{k'} \\ \bar{z}_{-k'} \end{pmatrix}_{\mathbf{2}_{k'}} &= (z_k z_{k'} + \bar{z}_{-k} \bar{z}_{-k'})_{\mathbf{1}_{+-}} \oplus (z_k z_{k'} - \bar{z}_{-k} \bar{z}_{-k'})_{\mathbf{1}_{-+}} \\ &\oplus \begin{pmatrix} z_k \bar{z}_{-k'} \\ (-1)^{kk'} \bar{z}_{-k} z_{k'} \end{pmatrix}_{\mathbf{2}_{k-k'}}, \end{aligned} \tag{179}$$

for $k + k' = N/2$ and $k - k' \neq 0$,

$$\begin{aligned} \begin{pmatrix} z_k \\ \bar{z}_{-k} \end{pmatrix}_{\mathbf{2}_k} \otimes \begin{pmatrix} z_{k'} \\ \bar{z}_{-k'} \end{pmatrix}_{\mathbf{2}_{k'}} &= \left(z_k \bar{z}_{-k'} + (-1)^{kk'} \bar{z}_{-k} z_{k'}\right)_{\mathbf{1}_{++}} \oplus \left(z_k \bar{z}_{-k'} - (-1)^{kk'} \bar{z}_{-k} z_{k'}\right)_{\mathbf{1}_{--}} \\ &\oplus \begin{pmatrix} z_k z_{k'} \\ (-1)^{kk'} \bar{z}_{-k} \bar{z}_{-k'} \end{pmatrix}_{\mathbf{2}_{k+k'}}, \end{aligned} \tag{180}$$

for $k + k' \neq N/2$ and $k - k' = 0$,

$$\begin{aligned} \begin{pmatrix} z_k \\ \bar{z}_{-k} \end{pmatrix}_{\mathbf{2}_k} \otimes \begin{pmatrix} z_{k'} \\ \bar{z}_{-k'} \end{pmatrix}_{\mathbf{2}_{k'}} &= \left(z_k \bar{z}_{-k'} + (-1)^{kk'} \bar{z}_{-k} z_{k'}\right)_{\mathbf{1}_{++}} \oplus \left(z_k \bar{z}_{-k'} - (-1)^{kk'} \bar{z}_{-k} z_{k'}\right)_{\mathbf{1}_{--}} \\ &\oplus (z_k z_{k'} + \bar{z}_{-k} \bar{z}_{-k'})_{\mathbf{1}_{+-}} \oplus (z_k z_{k'} - \bar{z}_{-k} \bar{z}_{-k'})_{\mathbf{1}_{-+}}, \end{aligned} \tag{181}$$

for $k + k' = N/2$ and $k - k' = 0$,

$$\begin{aligned} (w)_{\mathbf{1}_{++}} \otimes \begin{pmatrix} z_k \\ \bar{z}_{-k} \end{pmatrix}_{\mathbf{2}_k} &= \begin{pmatrix} w z_k \\ w \bar{z}_{-k} \end{pmatrix}_{\mathbf{2}_k}, \quad (w)_{\mathbf{1}_{--}} \otimes \begin{pmatrix} z_k \\ \bar{z}_{-k} \end{pmatrix}_{\mathbf{2}_k} = \begin{pmatrix} w z_k \\ -w \bar{z}_{-k} \end{pmatrix}_{\mathbf{2}_k}, \\ (w)_{\mathbf{1}_{+-}} \otimes \begin{pmatrix} z_k \\ \bar{z}_{-k} \end{pmatrix}_{\mathbf{2}_k} &= \begin{pmatrix} w \bar{z}_{-k} \\ w z_k \end{pmatrix}_{\mathbf{2}_k}, \quad (w)_{\mathbf{1}_{-+}} \otimes \begin{pmatrix} z_k \\ \bar{z}_{-k} \end{pmatrix}_{\mathbf{2}_k} = \begin{pmatrix} w \bar{z}_{-k} \\ -w z_k \end{pmatrix}_{\mathbf{2}_k}, \end{aligned} \tag{182}$$

$$\mathbf{1}_{s_1 s_2} \otimes \mathbf{1}_{s_1' s_2'} = \mathbf{1}_{s_1'' s_2''}, \tag{183}$$

where $s_1'' = s_1 s_1'$ and $s_2'' = s_2 s_2'$.

7.2 Q_4

Here we give simple examples. In this subsection, we show the results on Q_4 and in the next subsection we show Q_6.

	h	$\chi_{1_{++}}$	$\chi_{1_{+-}}$	$\chi_{1_{-+}}$	$\chi_{1_{--}}$	χ_2
C_1	1	1	1	1	1	2
C_2	4	1	-1	-1	1	$2\cos(\pi/2)$
C_1'	2	1	1	1	1	-2
C_2'	4	1	1	-1	-1	0
C_2''	4	1	-1	1	-1	0

Table 13: Characters of Q_4 representations

The Q_4 has the eight elements, $a^m b^k$, for $m = 0, 1, 2, 3$ and $k = 0, 1$, where a and b satisfy $a^4 = e$, $b^2 = a^2$ and $b^{-1}ab = a^{-1}$. These elements are classified into the five conjugacy classes,

$$\begin{array}{lcl} C_1: & \{e\}, & h=1, \\ C_2: & \{a, a^3\}, & h=4, \\ C_1': & \{a^2\}, & h=2, \\ C_2': & \{b, a^2 b\}, & h=4, \\ C_2'': & \{ab, a^3 b\}, & h=4, \end{array} \tag{184}$$

where we have also shown the orders of each element in the conjugacy class by h.

The Q_4 has four singlets, $\mathbf{1}_{++}$, $\mathbf{1}_{+-}$, $\mathbf{1}_{-+}$ and $\mathbf{1}_{--}$, and one doublet $\mathbf{2}$. The characters are shown in Table 13. The tensor products are obtained as

$$\begin{aligned} \begin{pmatrix} z \\ \bar{z} \end{pmatrix}_{\mathbf{2}} \otimes \begin{pmatrix} z' \\ \bar{z}' \end{pmatrix}_{\mathbf{2}} &= (z\bar{z}' - \bar{z}z')_{\mathbf{1}_{++}} \oplus (z\bar{z}' + \bar{z}z')_{\mathbf{1}_{--}} \\ &\oplus (zz' - \bar{z}\bar{z}')_{\mathbf{1}_{+-}} \oplus (zz' + \bar{z}\bar{z}')_{\mathbf{1}_{-+}}, \end{aligned} \tag{185}$$

$$\begin{aligned} &(w)_{\mathbf{1}_{++}} \otimes \begin{pmatrix} z \\ \bar{z} \end{pmatrix}_{\mathbf{2}} = \begin{pmatrix} wz \\ w\bar{z} \end{pmatrix}_{\mathbf{2}}, \quad (w)_{\mathbf{1}_{--}} \otimes \begin{pmatrix} z \\ \bar{z} \end{pmatrix}_{\mathbf{2}} = \begin{pmatrix} wz \\ -w\bar{z} \end{pmatrix}_{\mathbf{2}}, \\ &(w)_{\mathbf{1}_{+-}} \otimes \begin{pmatrix} z \\ \bar{z} \end{pmatrix}_{\mathbf{2}} = \begin{pmatrix} w\bar{z} \\ wz \end{pmatrix}_{\mathbf{2}}, \quad (w)_{\mathbf{1}_{-+}} \otimes \begin{pmatrix} z \\ \bar{z} \end{pmatrix}_{\mathbf{2}} = \begin{pmatrix} w\bar{z} \\ -wz \end{pmatrix}_{\mathbf{2}}, \end{aligned} \tag{186}$$

$$\mathbf{1}_{s_1 s_2} \otimes \mathbf{1}_{s_1' s_2'} = \mathbf{1}_{s_1'' s_2''}, \tag{187}$$

where $s_1'' = s_1 s_1'$ and $s_2'' = s_2 s_2'$. Note that some minus signs are different from the tensor products of D_4.

7.3 Q_6

The Q_6 has the 12 elements, $a^m b^k$, for $m = 0, 1, 2, 3, 4, 5$ and $k = 0, 1$, where a and b satisfy $a^6 = e$, $b^2 = a^3$ and $b^{-1}ab = a^{-1}$. These elements are classified into the six conjugacy

	h	$\chi_{1_{++}}$	$\chi_{1_{+-}}$	$\chi_{1_{-+}}$	$\chi_{1_{--}}$	χ_{2_1}	χ_{2_2}
C_1	1	1	1	1	1	2	2
C_2^1	6	1	-1	-1	1	$2\cos(2\pi/6)$	$2\cos(4\pi/6)$
C_2^2	3	1	1	1	1	$2\cos(4\pi/6)$	$2\cos(8\pi/6)$
C_1'	2	1	1	1	1	-2	2
C_3	4	1	i	$-i$	-1	0	0
C_3'	4	1	$-i$	i	-1	0	0

Table 14: Characters of Q_6 representations

classes,

$$\begin{array}{ccc} C_1: & \{e\}, & h=1, \\ C_2^{(1)}: & \{a, a^5\}, & h=6, \\ C_2^{(2)}: & \{a^2, a^4\}, & h=3, \\ C_1': & \{a^3\}, & h=2, \\ C_3: & \{b, a^2b, a^4b\}, & h=4, \\ C_3': & \{ab, a^3b, a^5b\}, & h=4. \end{array} \tag{188}$$

The Q_6 has four singlets, $\mathbf{1}_{++}$, $\mathbf{1}_{+-}$, $\mathbf{1}_{-+}$, and $\mathbf{1}_{--}$, and two doublets, $\mathbf{2}_1$ and $\mathbf{2}_2$. The characters are shown in Table 14. The tensor products are obtained as

$$\begin{pmatrix} z \\ \bar{z} \end{pmatrix}_{\mathbf{2}_2} \otimes \begin{pmatrix} z' \\ \bar{z}' \end{pmatrix}_{\mathbf{2}_1} = (zz' - \bar{z}\bar{z}')_{\mathbf{1}_{+-}} \oplus (zz' + \bar{z}\bar{z}')_{\mathbf{1}_{-+}} \oplus \begin{pmatrix} z\bar{z}' \\ \bar{z}z' \end{pmatrix}_{\mathbf{2}_1}, \tag{189}$$

$$\begin{pmatrix} z \\ \bar{z} \end{pmatrix}_{\mathbf{2}_k} \otimes \begin{pmatrix} z' \\ \bar{z}' \end{pmatrix}_{\mathbf{2}_k} = (z\bar{z}' - \bar{z}z')_{\mathbf{1}_{++}} \oplus (z\bar{z}' + \bar{z}z')_{\mathbf{1}_{--}} \oplus \begin{pmatrix} zz' \\ -\bar{z}\bar{z}' \end{pmatrix}_{\mathbf{2}_{k'}}, \tag{190}$$

for $k, k' = 1, 2$ and $k' \neq k$,

$$\begin{aligned} (w)_{\mathbf{1}_{++}} \otimes \begin{pmatrix} z_k \\ \bar{z}_{-k} \end{pmatrix}_{\mathbf{2}_k} &= \begin{pmatrix} wz_k \\ w\bar{z}_{-k} \end{pmatrix}_{\mathbf{2}_k}, \quad (w)_{\mathbf{1}_{--}} \otimes \begin{pmatrix} z_k \\ \bar{z}_{-k} \end{pmatrix}_{\mathbf{2}_k} = \begin{pmatrix} wz_k \\ -w\bar{z}_{-k} \end{pmatrix}_{\mathbf{2}_k}, \\ (w)_{\mathbf{1}_{+-}} \otimes \begin{pmatrix} z_k \\ \bar{z}_{-k} \end{pmatrix}_{\mathbf{2}_k} &= \begin{pmatrix} w\bar{z}_{-k} \\ wz_k \end{pmatrix}_{\mathbf{2}_k}, \quad (w)_{\mathbf{1}_{-+}} \otimes \begin{pmatrix} z_k \\ \bar{z}_{-k} \end{pmatrix}_{\mathbf{2}_k} = \begin{pmatrix} w\bar{z}_{-k} \\ -wz_k \end{pmatrix}_{\mathbf{2}_k}, \end{aligned} \tag{191}$$

$$\mathbf{1}_{s_1 s_2} \otimes \mathbf{1}_{s_1' s_2'} = \mathbf{1}_{s_1'' s_2''}, \tag{192}$$

where $s_1'' = s_1 s_1'$ and $s_2'' = s_2 s_2'$.

8 $\Sigma(2N^2)$

8.1 Generic aspects

The discrete group $\Sigma(2N^2)$ is isomorphic to $(Z_N \times Z'_N) \rtimes Z_2$. We denote the generators of Z_N and Z'_N by a and a', respectively, and the Z_2 generator is written by b. They satisfy

$$\begin{aligned} &a^N = a'^N = b^2 = e, \\ &aa' = a'a, \qquad bab = a'. \end{aligned} \tag{193}$$

Using them, all of $\Sigma(2N^2)$ elements are written as

$$g = b^k a^m a'^n, \tag{194}$$

for $k = 0, 1$ and $m, n = 0, 1, ..., N-1$.

These generators, a, a' and b, are represented, e.g. as

$$a = \begin{pmatrix} 1 & 0 \\ 0 & \rho \end{pmatrix}, \qquad a' = \begin{pmatrix} \rho & 0 \\ 0 & 1 \end{pmatrix}, \qquad b = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \tag{195}$$

where $\rho = e^{2\pi i/N}$. Then, all of $\Sigma(2N^2)$ elements are written as

$$\begin{pmatrix} \rho^m & 0 \\ 0 & \rho^n \end{pmatrix}, \qquad \begin{pmatrix} 0 & \rho^m \\ \rho^n & 0 \end{pmatrix}. \tag{196}$$

• Conjugacy classes

Now, let us study the conjugacy classes of $\Sigma(2N^2)$. We obtain the algebraic relations,

$$\begin{aligned} &b(a^l a'^m)b^{-1} = a^m a'^l, \qquad b(ba^l a'^m)b^{-1} = ba^m a'^l, \\ &a^k(ba^l a'^m)a^{-k} = ba^{l-k}a'^{m+k}, \qquad a'^k(ba^l a'^m)a'^{-k} = ba^{l+k}a'^{m-k}. \end{aligned} \tag{197}$$

Then, it is found that the $\Sigma(2N^2)$ group has the following conjugacy classes,

$$\begin{array}{ccc} C_1 : & \{e\}, & h = 1, \\ C_1^{(1)} : & \{aa'\}, & h = N, \\ \vdots & \vdots & \vdots \\ C_1^{(k)} : & \{a^k a'^k\}, & h = N/gcd(N,k), \\ \vdots & \vdots & \vdots \\ C_1^{(N-1)} : & \{a^{N-1}a'^{N-1}\}, & h = N/gcd(N,N-1), \\ C'^{(k)}_N : & \{ba^k, ba^{k-1}a', \cdots, ba'^k, \cdots, ba^{k+1}a'^{N-1}\}, & h = 2N/gcd(N,k), \\ C_2^{(l,m)} : & \{a^l a'^m \ , \ a'^l a^m\}, & h = N/gcd(N,l,m), \end{array} \tag{198}$$

where $l > m$ for $l, m = 0, \cdots, N-1$. The number of conjugacy classes $C_2^{(l,m)}$ is equal to $N(N-1)/2$. The total number of conjugacy classes of $\Sigma(2N^2)$ is equal to $N(N-1)/2 + N + N = (N^2 + 3N)/2$.

	h	$\chi_{1_{+n}}$	$\chi_{1_{-n}}$	$\chi_{2_{p,q}}$
C_1	1	1	1	2
$C_1^{(1)}$	N	ρ^{2n}	ρ^{2n}	$2\rho^{p+q}$
$\vdots$				
$C_1^{(N-1)}$	$N/gcd(N, N-1)$	$\rho^{2n(N-1)}$	$\rho^{2n(N-1)}$	$2\rho^{(N-1)(p+q)}$
$C_N'^{(k)}$	$2N/gcd(N,k)$	ρ^{kn}	$-\rho^{kn}$	0
$C_2^{(l,m)}$	$N/gcd(N,l,m)$	$\rho^{(l+m)n}$	$\rho^{(l+m)n}$	$\rho^{lq+mp}+\rho^{lp+mq}$

Table 15: Characters of $\Sigma(2N^2)$ representations

• **Characters and representations**

The orthogonality relations (12) and (13) for $\Sigma(2N^2)$ lead to

$$m_1 + 2^2 m_2 + \cdots = 2N^2, \tag{199}$$

$$m_1 + m_2 + \cdots = (N^2+3N)/2. \tag{200}$$

The solution is found as $(m_1, m_2) = (2N, N(N-1)/2)$. That is, there are $2N$ singlets and $N(N-1)/2$ doublets.

First of all, we study on singlets. Since a and a' belong to the same conjugacy class $C_2^{(1,0)}$, the characters $\chi_\alpha(g)$ for singlets should satisfy $\chi_\alpha(a) = \chi_\alpha(a')$. Furthermore, because of $b^2 = e$ and $a^N = e$, possible values of $\chi_\alpha(g)$ for singlets are found as $\chi_\alpha(a) = \rho^n$ and $\chi_\alpha(b) = \pm 1$. Then, totally we have $2N$ combinations, which correspond to $2N$ singlets, $\mathbf{1}_{\pm n}$ for $n = 0, 1, \cdots, N-1$. These characters are summarized in Table 15.

Next, we study doublet representations. Indeed, the matrices (195) correspond to a doublet representation. Similarly, (2×2) matrix representations for generic doublets $\mathbf{2}_{p,q}$ are written by replacing

$$a \to a^p a'^q \quad \text{and} \quad a' \to a^q a'^p. \tag{201}$$

That is, for doublets $\mathbf{2}_{p,q}$, the generators a and a' as well as b are represented as

$$a = \begin{pmatrix} \rho^q & 0 \\ 0 & \rho^p \end{pmatrix}, \qquad a' = \begin{pmatrix} \rho^p & 0 \\ 0 & \rho^q \end{pmatrix}, \qquad b = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \tag{202}$$

We denote the doublet $\mathbf{2}_{p,q}$ as

$$\mathbf{2}_{p,q} = \begin{pmatrix} x_q \\ x_p \end{pmatrix}, \tag{203}$$

where we take $p > q$ and $q = 0, 1, \cdots, N-2$. Then, each of up and down components, x_q and x_p, has definite $Z_N \times Z'_N$ charges. That is, x_q and x_p have $(q,0)$ and $(0,p)$ $Z_N \times Z'_N$ charges, respectively. The characters for doublets are also summarized in Table 15.

• **Tensor products**

Now, let us consider tensor products of doublets $\mathbf{2}_{p,q}$. Because of their $Z_N \times Z'_N$ charges, their tensor products can be obtained as

$$\begin{pmatrix} x_q \\ x_p \end{pmatrix}_{\mathbf{2}_{q,p}} \otimes \begin{pmatrix} y_{q'} \\ y_{p'} \end{pmatrix}_{\mathbf{2}_{q',p'}} = \begin{pmatrix} x_q y_{q'} \\ x_p y_{p'} \end{pmatrix}_{\mathbf{2}_{q+q',p+p'}} \oplus \begin{pmatrix} x_p y_{q'} \\ x_q y_{p'} \end{pmatrix}_{\mathbf{2}_{q'+p,q+p'}}, \tag{204}$$

for $q+q' \neq p+p'\ mod(N)$ and $q+p' \neq p+q'\ mod(N)$,

$$\begin{aligned} \begin{pmatrix} x_q \\ x_p \end{pmatrix}_{\mathbf{2}_{q,p}} \otimes \begin{pmatrix} y_{q'} \\ y_{p'} \end{pmatrix}_{\mathbf{2}_{q',p'}} &= (x_q y_{q'} + x_p y_{p'})_{\mathbf{1}_{+,q+q'}} \oplus (x_q y_{q'} - x_p y_{p'})_{\mathbf{1}_{-,q+q'}} \\ &\oplus \begin{pmatrix} x_p y_{q'} \\ x_q y_{p'} \end{pmatrix}_{\mathbf{2}_{q'+p,q+p'}}, \end{aligned} \tag{205}$$

for $q+q' = p+p'\ mod(N)$ and $q+p' \neq p+q'\ mod(N)$

$$\begin{aligned} \begin{pmatrix} x_q \\ x_p \end{pmatrix}_{\mathbf{2}_{q,p}} \otimes \begin{pmatrix} y_{q'} \\ y_{p'} \end{pmatrix}_{\mathbf{2}_{q',p'}} &= (x_p y_{q'} + x_q y_{p'})_{\mathbf{1}_{+,q+p'}} \oplus (x_p y_{q'} - x_q y_{p'})_{\mathbf{1}_{-,q+p'}} \\ &\oplus \begin{pmatrix} x_q y_{q'} \\ x_p y_{p'} \end{pmatrix}_{\mathbf{2}_{q+q',p+p'}}, \end{aligned} \tag{206}$$

for $q+q' \neq p+p'\ mod(N)$ and $q+p' = p+q'\ mod(N)$

$$\begin{aligned} \begin{pmatrix} x_q \\ x_p \end{pmatrix}_{\mathbf{2}_{q,p}} \otimes \begin{pmatrix} y_{q'} \\ y_{p'} \end{pmatrix}_{\mathbf{2}_{q',p'}} &= (x_q y_{q'} + x_p y_{p'})_{\mathbf{1}_{+,q+q'}} \oplus (x_q y_{q'} - x_p y_{p'})_{\mathbf{1}_{-,q+q'}} \\ &\oplus (x_p y_{q'} + x_q y_{p'})_{\mathbf{1}_{+,q+p'}} \oplus (x_p y_{q'} - x_q y_{p'})_{\mathbf{1}_{-,q+p'}}, \end{aligned} \tag{207}$$

for $q+q' = p+p'\ mod(N)$ and $q+p' = p+q'\ mod(N)$. In addition, the tensor products between singlets and doublets are obtained as

$$(y)_{\mathbf{1}_{s,n}} \otimes \begin{pmatrix} x_q \\ x_p \end{pmatrix}_{\mathbf{2}_{q,p}} = \begin{pmatrix} y x_q \\ y x_p \end{pmatrix}_{\mathbf{2}_{q+n,p+n}}. \tag{208}$$

These tensor products are independent of $s = \pm$. The tensor products of singlets are simply obtained as

$$\mathbf{1}_{sn} \otimes \mathbf{1}_{s'n'} = \mathbf{1}_{ss',n+n'}. \tag{209}$$

8.2 $\Sigma(18)$

The $\Sigma(2)$ group is nothing but the Abelian Z_2 group. Furthermore, the $\Sigma(8)$ group is isomorphic to D_4. Thus, the simple and non-trivial example is $\Sigma(18)$.

	h	$\chi_{1_{+0}}$	$\chi_{1_{+1}}$	$\chi_{1_{+2}}$	$\chi_{1_{-0}}$	$\chi_{1_{-1}}$	$\chi_{1_{-2}}$	$\chi_{2_{1,0}}$	$\chi_{2_{2,0}}$	$\chi_{2_{2,1}}$
C_1	1	1	1	1	1	1	1	2	2	2
$C_1^{(1)}$	3	1	ρ^2	ρ	1	ρ^2	ρ	2ρ	$2\rho^2$	2
$C_1^{(2)}$	3	1	ρ	ρ^2	1	ρ	ρ^2	$2\rho^2$	2ρ	2
$C_3'^{(0)}$	2	1	1	1	-1	-1	-1	0	0	0
$C_3'^{(1)}$	6	1	ρ	ρ^2	-1	$-\rho$	$-\rho^2$	0	0	0
$C_3'^{(2)}$	6	1	ρ^2	ρ	-1	$-\rho^2$	$-\rho$	0	0	0
$C_2^{(1,0)}$	3	1	ρ	ρ^2	1	ρ	ρ^2	$-\rho^2$	$-\rho$	-1
$C_1^{(2,0)}$	3	1	ρ^2	ρ	1	ρ^2	ρ	$-\rho$	$-\rho^2$	-1
$C_1^{(3,0)}$	3	1	1	1	1	1	1	-1	-1	-1

Table 16: Characters of $\Sigma(18)$ representations

The $\Sigma(18)$ has eighteen elements, $b^k a^m a'^n$ for $k = 0, 1$ and $m, n = 0, 1, 2$, where a, a' and b satisfy $b^2 = e$, $a^3 = a'^3 = e$, $aa' = a'a$ and $bab = a'$. These elements are classified into nine conjugacy classes,

$$\begin{array}{rccc}
C_1: & \{e\}, & h = 1, \\
C_1^{(1)}: & \{aa'\}, & h = 3, \\
C_1^{(2)}: & \{a^2 a'^2\}, & h = 3, \\
C_3'^{(0)}: & \{b\ ,\ ba'^2 a\ ,\ ba' a^2\}, & h = 2, \\
C_3'^{(1)}: & \{ba'\ ,\ ba\ ,\ ba'^2 a^2\}, & h = 6, \\
C_3'^{(2)}: & \{ba'^2\ ,\ ba' a\ ,\ ba^2\}, & h = 6, \\
C_2^{(1,0)}: & \{a\ ,\ a'\}, & h = 3, \\
C_2^{(2,0)}: & \{a^2\ ,\ a'^2\}, & h = 3, \\
C_2^{(2,1)}: & \{a^2 a'\ ,\ aa'^2\}, & h = 3,
\end{array} \tag{210}$$

where we have also shown the orders of each element in the conjugacy class by h.

The $\Sigma(18)$ has six singlets $\mathbf{1}_{\pm,n}$ with $n = 0, 1, 2$ and three doublets $\mathbf{2}_{p,q}$ with $(p, q) = (1, 0), (2, 0), (2, 1)$. The characters are shown in Table 16.

The tensor products between doublets are obtained as

$$
\begin{aligned}
&\begin{pmatrix} x_2 \\ x_1 \end{pmatrix}_{\mathbf{2}_{2,1}} \otimes \begin{pmatrix} y_2 \\ y_1 \end{pmatrix}_{\mathbf{2}_{2,1}} = (x_1y_2 + x_2y_1)_{\mathbf{1}_{+,0}} \oplus (x_1y_2 - x_2y_1)_{\mathbf{1}_{-,0}} \oplus \begin{pmatrix} x_1y_1 \\ x_2y_2 \end{pmatrix}_{\mathbf{2}_{2,1}} , \\
&\begin{pmatrix} x_2 \\ x_0 \end{pmatrix}_{\mathbf{2}_{2,0}} \otimes \begin{pmatrix} y_2 \\ y_0 \end{pmatrix}_{\mathbf{2}_{2,0}} = (x_0y_2 + x_2y_0)_{\mathbf{1}_{+,2}} \oplus (x_0y_2 - x_2y_0)_{\mathbf{1}_{-,2}} \oplus \begin{pmatrix} x_2y_2 \\ x_0y_0 \end{pmatrix}_{\mathbf{2}_{1,0}} , \\
&\begin{pmatrix} x_1 \\ x_0 \end{pmatrix}_{\mathbf{2}_{1,0}} \otimes \begin{pmatrix} y_1 \\ y_0 \end{pmatrix}_{\mathbf{2}_{1,0}} = (x_0y_1 + x_1y_0)_{\mathbf{1}_{+,1}} \oplus (x_0y_1 - x_1y_0)_{\mathbf{1}_{-,1}} \oplus \begin{pmatrix} x_1y_1 \\ x_0y_0 \end{pmatrix}_{\mathbf{2}_{2,0}} , \\
&\begin{pmatrix} x_2 \\ x_1 \end{pmatrix}_{\mathbf{2}_{2,1}} \otimes \begin{pmatrix} y_2 \\ y_0 \end{pmatrix}_{\mathbf{2}_{2,0}} = (x_2y_2 + x_1y_0)_{\mathbf{1}_{+,1}} \oplus (x_2y_2 - x_1y_0)_{\mathbf{1}_{-,1}} \oplus \begin{pmatrix} x_2y_0 \\ x_1y_2 \end{pmatrix}_{\mathbf{2}_{2,0}} , \\
&\begin{pmatrix} x_2 \\ x_1 \end{pmatrix}_{\mathbf{2}_{2,1}} \otimes \begin{pmatrix} y_1 \\ y_0 \end{pmatrix}_{\mathbf{2}_{1,0}} = (x_1y_1 + x_2y_0)_{\mathbf{1}_{+,2}} \oplus (x_1y_1 - x_2y_0)_{\mathbf{1}_{-,2}} \oplus \begin{pmatrix} x_1y_0 \\ x_2y_1 \end{pmatrix}_{\mathbf{2}_{1,0}} , \\
&\begin{pmatrix} x_2 \\ x_0 \end{pmatrix}_{\mathbf{2}_{2,0}} \otimes \begin{pmatrix} y_1 \\ y_0 \end{pmatrix}_{\mathbf{2}_{1,0}} = (x_2y_1 + x_0y_0)_{\mathbf{1}_{+,0}} \oplus (x_2y_1 - x_0y_0)_{\mathbf{1}_{-,0}} \oplus \begin{pmatrix} x_2y_0 \\ x_0y_1 \end{pmatrix}_{\mathbf{2}_{2,1}} .
\end{aligned}
\tag{211}
$$

The tensor products between singlets are obtained as

$$
\begin{aligned}
&\mathbf{1}_{\pm,0} \otimes \mathbf{1}_{\pm,0} = \mathbf{1}_{+,0} \ , \ \mathbf{1}_{\pm,1} \otimes \mathbf{1}_{\pm,1} = \mathbf{1}_{+,2} \ , \ \mathbf{1}_{\pm,2} \otimes \mathbf{1}_{\pm,2} = \mathbf{1}_{+,1} \ , \ \mathbf{1}_{\pm,1} \otimes \mathbf{1}_{\pm,0} = \mathbf{1}_{+,1} \ , \\
&\mathbf{1}_{\pm,2} \otimes \mathbf{1}_{\pm,0} = \mathbf{1}_{+,2} \ , \ \mathbf{1}_{\pm,2} \otimes \mathbf{1}_{\pm,1} = \mathbf{1}_{+,0} \ , \ \mathbf{1}_{\pm,0} \otimes \mathbf{1}_{\mp,0} = \mathbf{1}_{-,0} \ , \ \mathbf{1}_{\pm,1} \otimes \mathbf{1}_{\mp,1} = \mathbf{1}_{-,2} \ , \\
&\mathbf{1}_{\pm,2} \otimes \mathbf{1}_{\mp,2} = \mathbf{1}_{-,1} \ , \ \mathbf{1}_{\pm,1} \otimes \mathbf{1}_{\mp,0} = \mathbf{1}_{-,1} \ , \ \mathbf{1}_{\pm,2} \otimes \mathbf{1}_{\pm,0} = \mathbf{1}_{-,2} \ , \ \mathbf{1}_{\pm,2} \otimes \mathbf{1}_{\mp,1} = \mathbf{1}_{-,0} \ .
\end{aligned}
\tag{212}
$$

The tensor products between singlets and doublets are obtained as

$$
\begin{aligned}
&(y)_{\mathbf{1}_{\pm,0}} \otimes \begin{pmatrix} x_2 \\ x_1 \end{pmatrix}_{\mathbf{2}_{2,1}} = \begin{pmatrix} yx_2 \\ yx_1 \end{pmatrix}_{\mathbf{2}_{2,1}} , \ (y)_{\mathbf{1}_{\pm,1}} \otimes \begin{pmatrix} x_2 \\ x_1 \end{pmatrix}_{\mathbf{2}_{2,1}} = \begin{pmatrix} yx_1 \\ yx_2 \end{pmatrix}_{\mathbf{2}_{2,0}} , \\
&(y)_{\mathbf{1}_{\pm,2}} \otimes \begin{pmatrix} x_2 \\ x_1 \end{pmatrix}_{\mathbf{2}_{2,1}} = \begin{pmatrix} yx_2 \\ yx_1 \end{pmatrix}_{\mathbf{2}_{1,0}} , \ (y)_{\mathbf{1}_{\pm,0}} \otimes \begin{pmatrix} x_2 \\ x_0 \end{pmatrix}_{\mathbf{2}_{2,0}} = \begin{pmatrix} yx_2 \\ yx_0 \end{pmatrix}_{\mathbf{2}_{2,0}} , \\
&(y)_{\mathbf{1}_{\pm,1}} \otimes \begin{pmatrix} x_2 \\ x_0 \end{pmatrix}_{\mathbf{2}_{2,0}} = \begin{pmatrix} yx_0 \\ yx_2 \end{pmatrix}_{\mathbf{2}_{1,0}} , \ (y)_{\mathbf{1}_{\pm,2}} \otimes \begin{pmatrix} x_2 \\ x_0 \end{pmatrix}_{\mathbf{2}_{2,0}} = \begin{pmatrix} yx_0 \\ yx_2 \end{pmatrix}_{\mathbf{2}_{2,1}} , \\
&(y)_{\mathbf{1}_{\pm,0}} \otimes \begin{pmatrix} x_1 \\ x_0 \end{pmatrix}_{\mathbf{2}_{1,0}} = \begin{pmatrix} yx_1 \\ yx_0 \end{pmatrix}_{\mathbf{2}_{1,0}} , \ (y)_{\mathbf{1}_{\pm,1}} \otimes \begin{pmatrix} x_1 \\ x_0 \end{pmatrix}_{\mathbf{2}_{1,0}} = \begin{pmatrix} yx_1 \\ yx_0 \end{pmatrix}_{\mathbf{2}_{2,1}} , \\
&(y)_{\mathbf{1}_{\pm,2}} \otimes \begin{pmatrix} x_1 \\ x_0 \end{pmatrix}_{\mathbf{2}_{1,0}} = \begin{pmatrix} yx_0 \\ yx_1 \end{pmatrix}_{\mathbf{2}_{2,0}} .
\end{aligned}
\tag{213}
$$

8.3 $\Sigma(32)$

The $\Sigma(32)$ has thirty-two elements, $b^k a^m a'^n$ for $k = 0, 1$ and $m, n = 0, 1, 2, 3$, where a, a' and b satisfy $b^2 = e$, $a^4 = a'^4 = e$, $aa' = a'a$ and $bab = a'$. These elements are classified

	h	$\chi_{1_{+0}}$	$\chi_{1_{+1}}$	$\chi_{1_{+2}}$	$\chi_{1_{+3}}$	$\chi_{1_{-0}}$	$\chi_{1_{-1}}$	$\chi_{1_{-2}}$	$\chi_{1_{-3}}$	$\chi_{2_{1,0}}$	$\chi_{2_{2,0}}$	$\chi_{2_{2,1}}$	$\chi_{2_{3,0}}$	$\chi_{2_{3,1}}$	$\chi_{2_{3,2}}$
C_1	1	1	1	1	1	1	1	1	1	2	2	2	2	2	2
$C_1^{(1)}$	4	1	-1	1	-1	1	-1	1	-1	$2i$	-2	$-2i$	$-2i$	2	$2i$
$C_1^{(2)}$	2	1	1	1	1	1	1	1	1	-2	2	-2	-2	2	-2
$C_1^{(3)}$	4	1	-1	1	-1	1	-1	1	-1	$-2i$	-2	$2i$	$2i$	2	$-2i$
$C_4'^{(0)}$	2	1	1	1	1	-1	-1	-1	-1	0	0	0	0	0	0
$C_4'^{(1)}$	8	1	i	-1	$-i$	-1	$-i$	1	i	0	0	0	0	0	0
$C_4'^{(2)}$	4	1	-1	1	-1	-1	1	-1	1	0	0	0	0	0	0
$C_4'^{(3)}$	8	1	$-i$	-1	i	-1	i	1	$-i$	0	0	0	0	0	0
$C_2^{(1,0)}$	4	1	i	-1	$-i$	1	i	-1	$-i$	$1+i$	0	$-1+i$	$1-i$	0	$-1-i$
$C_2^{(2,0)}$	2	1	-1	1	-1	1	-1	1	-1	0	2	0	0	-2	0
$C_2^{(2,1)}$	4	1	$-i$	-1	i	1	$-i$	-1	i	$-1+i$	0	$1+i$	$-1-i$	0	$1-i$
$C_2^{(3,0)}$	4	1	$-i$	-1	i	1	$-i$	-1	i	$1-i$	0	$-1-i$	$1+i$	0	$-1+i$
$C_2^{(3,1)}$	4	1	1	1	1	1	1	1	1	0	-2	0	0	-2	0
$C_2^{(3,2)}$	4	1	i	-1	$-i$	1	i	-1	$-i$	$-1-i$	0	$1-i$	$-1+i$	0	$1+i$

Table 17: Characters of $\Sigma(32)$ representations

into fourteen conjugacy classes,

$$\begin{array}{rcl} C_1: & \{e\}, & h=1, \\ C_1^{(1)}: & \{aa'\}, & h=4, \\ C_1^{(2)}: & \{a^2a'^2\}, & h=2, \\ C_1^{(3)}: & \{a^3a'^3\}, & h=4, \\ C_4'^{(0)}: & \{b\ ,\ ba'a^3\ ,\ ba'^2a^2\ ,\ ba'^3a\}, & h=2, \\ C_4'^{(1)}: & \{ba'\ ,\ ba\ ,\ ba'^2a^3\ ,\ ba'^3a^2\}, & h=8, \\ C_4'^{(2)}: & \{ba'^2\ ,\ ba'a\ ,\ ba^2\ ,\ ba'^3a^3\}, & h=4, \\ C_4'^{(3)}: & \{ba'^3\ ,\ ba'^2a\ ,\ ba'a^2\ ,\ ba^3\}, & h=8, \\ C_2^{(1,0)}: & \{a\ ,\ a'\}, & h=4, \\ C_2^{(2,0)}: & \{a^2\ ,\ a'^2\}, & h=2, \\ C_2^{(2,1)}: & \{a^2a'\ ,\ aa'^2\}, & h=4, \\ C_2^{(3,0)}: & \{a^3\ ,\ a'^3\}, & h=4, \\ C_2^{(3,1)}: & \{a^3a'\ ,\ aa'^3\}, & h=4, \\ C_2^{(3,2)}: & \{a^3a'^2\ ,\ a^2a'^3\}, & h=4, \end{array} \tag{214}$$

where we have also shown the orders of each element in the conjugacy class by h.

The $\Sigma(32)$ has eight singlets $\mathbf{1}_{\pm,n}$ with $n=0,1,2,3$ and six doublets $\mathbf{2}_{p,q}$ with $(p,q)=(1,0),(2,0),(3,0),(2,1),(3,1),(3,2)$. The characters are shown in Table 17.

The tensor products between doublets are obtained as

$$\begin{pmatrix} x_3 \\ x_2 \end{pmatrix}_{\mathbf{2}_{3,2}} \otimes \begin{pmatrix} y_3 \\ y_2 \end{pmatrix}_{\mathbf{2}_{3,2}} = (x_2y_3 + x_3y_2)_{\mathbf{1}_{+,1}} \oplus (x_2y_3 - x_3y_2)_{\mathbf{1}_{-,1}} \oplus \begin{pmatrix} x_3y_3 \\ x_2y_2 \end{pmatrix}_{\mathbf{2}_{2,0}}, \quad (215)$$

$$\begin{aligned} \begin{pmatrix} x_3 \\ x_1 \end{pmatrix}_{\mathbf{2}_{3,1}} \otimes \begin{pmatrix} y_3 \\ y_1 \end{pmatrix}_{\mathbf{2}_{3,1}} &= (x_1y_3 + x_3y_1)_{\mathbf{1}_{+,0}} \oplus (x_1y_3 - x_3y_1)_{\mathbf{1}_{-,0}} \\ &\oplus (x_3y_3 + x_1y_1)_{\mathbf{1}_{+,2}} \oplus (x_3y_3 - x_1y_1)_{\mathbf{1}_{-,2}}, \end{aligned} \quad (216)$$

$$\begin{pmatrix} x_3 \\ x_0 \end{pmatrix}_{\mathbf{2}_{3,0}} \otimes \begin{pmatrix} y_3 \\ y_0 \end{pmatrix}_{\mathbf{2}_{3,0}} = (x_0y_3 + x_3y_0)_{\mathbf{1}_{+,3}} \oplus (x_0y_3 - x_3y_0)_{\mathbf{1}_{-,3}} \oplus \begin{pmatrix} x_3y_3 \\ x_0y_0 \end{pmatrix}_{\mathbf{2}_{2,0}}, \quad (217)$$

$$\begin{pmatrix} x_2 \\ x_1 \end{pmatrix}_{\mathbf{2}_{2,1}} \otimes \begin{pmatrix} y_2 \\ y_1 \end{pmatrix}_{\mathbf{2}_{2,1}} = (x_1y_2 + x_2y_1)_{\mathbf{1}_{+,3}} \oplus (x_1y_2 - x_2y_1)_{\mathbf{1}_{-,3}} \oplus \begin{pmatrix} x_1y_1 \\ x_2y_2 \end{pmatrix}_{\mathbf{2}_{2,0}}, \quad (218)$$

$$\begin{aligned} \begin{pmatrix} x_2 \\ x_0 \end{pmatrix}_{\mathbf{2}_{2,0}} \otimes \begin{pmatrix} y_2 \\ y_0 \end{pmatrix}_{\mathbf{2}_{2,0}} &= (x_0y_2 + x_2y_0)_{\mathbf{1}_{+,2}} \oplus (x_0y_2 - x_2y_0)_{\mathbf{1}_{-,2}} \\ &\oplus (x_2y_2 + x_0y_0)_{\mathbf{1}_{+,0}} \oplus (x_2y_2 - x_0y_0)_{\mathbf{1}_{-,0}}, \end{aligned} \quad (219)$$

$$\begin{pmatrix} x_1 \\ x_0 \end{pmatrix}_{\mathbf{2}_{1,0}} \otimes \begin{pmatrix} y_1 \\ y_0 \end{pmatrix}_{\mathbf{2}_{1,0}} = (x_0y_1 + x_1y_0)_{\mathbf{1}_{+,1}} \oplus (x_0y_1 - x_1y_0)_{\mathbf{1}_{-,1}} \oplus \begin{pmatrix} x_1y_1 \\ x_0y_0 \end{pmatrix}_{\mathbf{2}_{2,0}}, \quad (220)$$

$$\begin{pmatrix} x_3 \\ x_2 \end{pmatrix}_{\mathbf{2}_{3,2}} \otimes \begin{pmatrix} y_3 \\ y_1 \end{pmatrix}_{\mathbf{2}_{3,1}} = \begin{pmatrix} x_2y_3 \\ x_3y_1 \end{pmatrix}_{\mathbf{2}_{1,0}} \oplus \begin{pmatrix} x_2y_1 \\ x_3y_3 \end{pmatrix}_{\mathbf{2}_{3,2}}, \quad (221)$$

$$\begin{pmatrix} x_3 \\ x_2 \end{pmatrix}_{\mathbf{2}_{3,2}} \otimes \begin{pmatrix} y_3 \\ y_0 \end{pmatrix}_{\mathbf{2}_{3,0}} = (x_3y_3 + x_2y_0)_{\mathbf{1}_{+,2}} \oplus (x_3y_3 - x_2y_0)_{\mathbf{1}_{-,2}} \oplus \begin{pmatrix} x_3y_0 \\ x_2y_3 \end{pmatrix}_{\mathbf{2}_{3,1}}, \quad (222)$$

$$\begin{pmatrix} x_3 \\ x_2 \end{pmatrix}_{\mathbf{2}_{3,2}} \otimes \begin{pmatrix} y_2 \\ y_1 \end{pmatrix}_{\mathbf{2}_{2,1}} = (x_2y_2 + x_3y_1)_{\mathbf{1}_{+,0}} \oplus (x_2y_2 - x_3y_1)_{\mathbf{1}_{-,0}} \oplus \begin{pmatrix} x_2y_1 \\ x_3y_2 \end{pmatrix}_{\mathbf{2}_{3,1}}, \quad (223)$$

$$\begin{pmatrix} x_3 \\ x_2 \end{pmatrix}_{\mathbf{2}_{3,2}} \otimes \begin{pmatrix} y_2 \\ y_0 \end{pmatrix}_{\mathbf{2}_{2,0}} = \begin{pmatrix} x_3y_0 \\ x_2y_2 \end{pmatrix}_{\mathbf{2}_{3,1}} \oplus \begin{pmatrix} x_2y_0 \\ x_3y_2 \end{pmatrix}_{\mathbf{2}_{2,1}}, \quad (224)$$

$$\begin{pmatrix} x_3 \\ x_2 \end{pmatrix}_{\mathbf{2}_{3,2}} \otimes \begin{pmatrix} y_1 \\ y_0 \end{pmatrix}_{\mathbf{2}_{1,0}} = (x_2y_1 + x_3y_0)_{\mathbf{1}_{+,3}} \oplus (x_2y_1 - x_3y_0)_{\mathbf{1}_{-,3}} \oplus \begin{pmatrix} x_2y_0 \\ x_3y_1 \end{pmatrix}_{\mathbf{2}_{2,0}}, \quad (225)$$

$$\begin{pmatrix} x_3 \\ x_1 \end{pmatrix}_{\mathbf{2}_{3,1}} \otimes \begin{pmatrix} y_3 \\ y_0 \end{pmatrix}_{\mathbf{2}_{3,0}} = \begin{pmatrix} x_3y_0 \\ x_1y_3 \end{pmatrix}_{\mathbf{2}_{3,0}} \oplus \begin{pmatrix} x_3y_3 \\ x_1y_0 \end{pmatrix}_{\mathbf{2}_{2,1}}, \quad (226)$$

$$\begin{pmatrix} x_3 \\ x_1 \end{pmatrix}_{\mathbf{2}_{3,1}} \otimes \begin{pmatrix} y_2 \\ y_1 \end{pmatrix}_{\mathbf{2}_{2,1}} = \begin{pmatrix} x_1y_2 \\ x_3y_1 \end{pmatrix}_{\mathbf{2}_{3,0}} \oplus \begin{pmatrix} x_1y_1 \\ x_3y_2 \end{pmatrix}_{\mathbf{2}_{2,1}}, \quad (227)$$

$$\begin{pmatrix} x_3 \\ x_1 \end{pmatrix}_{\mathbf{2}_{3,1}} \otimes \begin{pmatrix} y_2 \\ y_0 \end{pmatrix}_{\mathbf{2}_{2,0}} = (x_1y_2 + x_3y_0)_{\mathbf{1}_{+,3}} \oplus (x_1y_2 - x_3y_0)_{\mathbf{1}_{-,3}} \oplus (x_3y_2 + x_1y_0)_{\mathbf{1}_{+,1}} \oplus (x_3y_2 - x_1y_0)_{\mathbf{1}_{-,1}}, \tag{228}$$

$$\begin{pmatrix} x_3 \\ x_1 \end{pmatrix}_{\mathbf{2}_{3,1}} \otimes \begin{pmatrix} y_1 \\ y_0 \end{pmatrix}_{\mathbf{2}_{1,0}} = \begin{pmatrix} x_3y_0 \\ x_1y_1 \end{pmatrix}_{\mathbf{2}_{3,2}} \oplus \begin{pmatrix} x_1y_0 \\ x_3y_1 \end{pmatrix}_{\mathbf{2}_{1,0}}, \tag{229}$$

$$\begin{pmatrix} x_3 \\ x_0 \end{pmatrix}_{\mathbf{2}_{3,0}} \otimes \begin{pmatrix} y_2 \\ y_1 \end{pmatrix}_{\mathbf{2}_{2,1}} = (x_3y_2 + x_0y_1)_{\mathbf{1}_{+,1}} \oplus (x_3y_2 - x_0y_1)_{\mathbf{1}_{-,1}} \oplus \begin{pmatrix} x_0y_2 \\ x_3y_1 \end{pmatrix}_{\mathbf{2}_{2,0}}, \tag{230}$$

$$\begin{pmatrix} x_3 \\ x_0 \end{pmatrix}_{\mathbf{2}_{3,0}} \otimes \begin{pmatrix} y_2 \\ y_0 \end{pmatrix}_{\mathbf{2}_{2,0}} = \begin{pmatrix} x_3y_0 \\ x_0y_2 \end{pmatrix}_{\mathbf{2}_{3,2}} \oplus \begin{pmatrix} x_3y_2 \\ x_0y_0 \end{pmatrix}_{\mathbf{2}_{1,0}}, \tag{231}$$

$$\begin{pmatrix} x_3 \\ x_0 \end{pmatrix}_{\mathbf{2}_{3,0}} \otimes \begin{pmatrix} y_1 \\ y_0 \end{pmatrix}_{\mathbf{2}_{1,0}} = (x_3y_1 + x_0y_0)_{\mathbf{1}_{+,0}} \oplus (x_3y_1 - x_0y_0)_{\mathbf{1}_{-,0}} \oplus \begin{pmatrix} x_3y_0 \\ x_0y_1 \end{pmatrix}_{\mathbf{2}_{3,1}}, \tag{232}$$

$$\begin{pmatrix} x_2 \\ x_1 \end{pmatrix}_{\mathbf{2}_{2,1}} \otimes \begin{pmatrix} y_2 \\ y_0 \end{pmatrix}_{\mathbf{2}_{2,0}} = (x_2y_2 + x_1y_0)_{\mathbf{1}_{+,1}} \oplus (x_2y_2 - x_1y_0)_{\mathbf{1}_{-,1}} \oplus \begin{pmatrix} x_1y_2 \\ x_2y_0 \end{pmatrix}_{\mathbf{2}_{3,2}}, \tag{233}$$

$$\begin{pmatrix} x_2 \\ x_1 \end{pmatrix}_{\mathbf{2}_{2,1}} \otimes \begin{pmatrix} y_1 \\ y_0 \end{pmatrix}_{\mathbf{2}_{1,0}} = (x_1y_1 + x_2y_0)_{\mathbf{1}_{+,2}} \oplus (x_1y_1 - x_2y_0)_{\mathbf{1}_{-,2}} \oplus \begin{pmatrix} x_2y_1 \\ x_1y_0 \end{pmatrix}_{\mathbf{2}_{3,1}}, \tag{234}$$

$$\begin{pmatrix} x_2 \\ x_0 \end{pmatrix}_{\mathbf{2}_{2,0}} \otimes \begin{pmatrix} y_1 \\ y_0 \end{pmatrix}_{\mathbf{2}_{1,0}} = \begin{pmatrix} x_2y_1 \\ x_0y_0 \end{pmatrix}_{\mathbf{2}_{2,0}} \oplus \begin{pmatrix} x_2y_0 \\ x_0y_1 \end{pmatrix}_{\mathbf{2}_{2,1}}. \tag{235}$$

The tensor products between singlets are obtained as

$$\begin{aligned}
&\mathbf{1}_{\pm,0} \otimes \mathbf{1}_{\pm,0} = \mathbf{1}_{+,0}\ ,\ \mathbf{1}_{\pm,1} \otimes \mathbf{1}_{\pm,1} = \mathbf{1}_{+,2}\ ,\ \mathbf{1}_{\pm,2} \otimes \mathbf{1}_{\pm,2} = \mathbf{1}_{+,0}\ ,\ \mathbf{1}_{\pm,3} \otimes \mathbf{1}_{\pm,3} = \mathbf{1}_{+,2},\\
&\mathbf{1}_{\pm,3} \otimes \mathbf{1}_{\pm,2} = \mathbf{1}_{+,1}\ ,\ \mathbf{1}_{\pm,3} \otimes \mathbf{1}_{\pm,1} = \mathbf{1}_{+,0}\ ,\ \mathbf{1}_{\pm,3} \otimes \mathbf{1}_{\pm,0} = \mathbf{1}_{+,3}\ ,\ \mathbf{1}_{\pm,2} \otimes \mathbf{1}_{\pm,1} = \mathbf{1}_{+,3},\\
&\mathbf{1}_{\pm,2} \otimes \mathbf{1}_{\pm,0} = \mathbf{1}_{+,2}\ ,\ \mathbf{1}_{\pm,1} \otimes \mathbf{1}_{\pm,0} = \mathbf{1}_{+,1},\\
&\mathbf{1}_{\mp,0} \otimes \mathbf{1}_{\pm,0} = \mathbf{1}_{-,0}\ ,\ \mathbf{1}_{\mp,1} \otimes \mathbf{1}_{\pm,1} = \mathbf{1}_{-,2}\ ,\ \mathbf{1}_{\mp,2} \otimes \mathbf{1}_{\pm,2} = \mathbf{1}_{-,0}\ ,\ \mathbf{1}_{\mp,3} \otimes \mathbf{1}_{\pm,3} = \mathbf{1}_{-,2},\\
&\mathbf{1}_{\mp,3} \otimes \mathbf{1}_{\pm,2} = \mathbf{1}_{-,1}\ ,\ \mathbf{1}_{\mp,3} \otimes \mathbf{1}_{\pm,1} = \mathbf{1}_{-,0}\ ,\ \mathbf{1}_{\mp,3} \otimes \mathbf{1}_{\pm,0} = \mathbf{1}_{-,3}\ ,\ \mathbf{1}_{\mp,2} \otimes \mathbf{1}_{\pm,1} = \mathbf{1}_{-,3},\\
&\mathbf{1}_{\mp,2} \otimes \mathbf{1}_{\pm,0} = \mathbf{1}_{-,2}\ ,\ \mathbf{1}_{\mp,1} \otimes \mathbf{1}_{\pm,0} = \mathbf{1}_{-,1}.
\end{aligned} \tag{236}$$

The tensor products between singlets and doublets are obtained as

$$
\begin{aligned}
&(y)_{\mathbf{1}_{\pm,0}} \otimes \begin{pmatrix} x_3 \\ x_2 \end{pmatrix}_{\mathbf{2}_{3,2}} = \begin{pmatrix} yx_3 \\ yx_2 \end{pmatrix}_{\mathbf{2}_{3,2}}, \quad (y)_{\mathbf{1}_{\pm,1}} \otimes \begin{pmatrix} x_3 \\ x_2 \end{pmatrix}_{\mathbf{2}_{3,2}} = \begin{pmatrix} yx_2 \\ yx_3 \end{pmatrix}_{\mathbf{2}_{3,0}}, \\
&(y)_{\mathbf{1}_{\pm,2}} \otimes \begin{pmatrix} x_3 \\ x_2 \end{pmatrix}_{\mathbf{2}_{3,2}} = \begin{pmatrix} yx_3 \\ yx_2 \end{pmatrix}_{\mathbf{2}_{1,0}}, \quad (y)_{\mathbf{1}_{\pm,3}} \otimes \begin{pmatrix} x_3 \\ x_2 \end{pmatrix}_{\mathbf{2}_{3,2}} = \begin{pmatrix} yx_3 \\ yx_2 \end{pmatrix}_{\mathbf{2}_{2,1}}, \\
&(y)_{\mathbf{1}_{\pm,0}} \otimes \begin{pmatrix} x_3 \\ x_1 \end{pmatrix}_{\mathbf{2}_{3,1}} = \begin{pmatrix} yx_3 \\ yx_1 \end{pmatrix}_{\mathbf{2}_{3,1}}, \quad (y)_{\mathbf{1}_{\pm,1}} \otimes \begin{pmatrix} x_3 \\ x_1 \end{pmatrix}_{\mathbf{2}_{3,1}} = \begin{pmatrix} yx_1 \\ yx_3 \end{pmatrix}_{\mathbf{2}_{2,0}}, \\
&(y)_{\mathbf{1}_{\pm,2}} \otimes \begin{pmatrix} x_3 \\ x_1 \end{pmatrix}_{\mathbf{2}_{3,1}} = \begin{pmatrix} yx_1 \\ yx_3 \end{pmatrix}_{\mathbf{2}_{3,1}}, \quad (y)_{\mathbf{1}_{\pm,3}} \otimes \begin{pmatrix} x_3 \\ x_1 \end{pmatrix}_{\mathbf{2}_{3,1}} = \begin{pmatrix} yx_3 \\ yx_1 \end{pmatrix}_{\mathbf{2}_{2,0}}, \\
&(y)_{\mathbf{1}_{\pm,0}} \otimes \begin{pmatrix} x_3 \\ x_0 \end{pmatrix}_{\mathbf{2}_{3,0}} = \begin{pmatrix} yx_3 \\ yx_0 \end{pmatrix}_{\mathbf{2}_{3,0}}, \quad (y)_{\mathbf{1}_{\pm,1}} \otimes \begin{pmatrix} x_3 \\ x_0 \end{pmatrix}_{\mathbf{2}_{3,0}} = \begin{pmatrix} yx_0 \\ yx_3 \end{pmatrix}_{\mathbf{2}_{1,0}}, \\
&(y)_{\mathbf{1}_{\pm,2}} \otimes \begin{pmatrix} x_3 \\ x_0 \end{pmatrix}_{\mathbf{2}_{3,0}} = \begin{pmatrix} yx_0 \\ yx_3 \end{pmatrix}_{\mathbf{2}_{2,1}}, \quad (y)_{\mathbf{1}_{\pm,3}} \otimes \begin{pmatrix} x_3 \\ x_0 \end{pmatrix}_{\mathbf{2}_{3,0}} = \begin{pmatrix} yx_0 \\ yx_3 \end{pmatrix}_{\mathbf{2}_{3,2}}, \\
&(y)_{\mathbf{1}_{\pm,0}} \otimes \begin{pmatrix} x_2 \\ x_1 \end{pmatrix}_{\mathbf{2}_{2,1}} = \begin{pmatrix} yx_2 \\ yx_1 \end{pmatrix}_{\mathbf{2}_{2,1}}, \quad (y)_{\mathbf{1}_{\pm,1}} \otimes \begin{pmatrix} x_2 \\ x_1 \end{pmatrix}_{\mathbf{2}_{2,1}} = \begin{pmatrix} yx_2 \\ yx_1 \end{pmatrix}_{\mathbf{2}_{3,2}}, \\
&(y)_{\mathbf{1}_{\pm,2}} \otimes \begin{pmatrix} x_2 \\ x_1 \end{pmatrix}_{\mathbf{2}_{2,1}} = \begin{pmatrix} yx_1 \\ yx_2 \end{pmatrix}_{\mathbf{2}_{3,0}}, \quad (y)_{\mathbf{1}_{\pm,3}} \otimes \begin{pmatrix} x_2 \\ x_1 \end{pmatrix}_{\mathbf{2}_{2,1}} = \begin{pmatrix} yx_2 \\ yx_1 \end{pmatrix}_{\mathbf{2}_{1,0}}, \\
&(y)_{\mathbf{1}_{\pm,0}} \otimes \begin{pmatrix} x_2 \\ x_0 \end{pmatrix}_{\mathbf{2}_{2,0}} = \begin{pmatrix} yx_2 \\ yx_0 \end{pmatrix}_{\mathbf{2}_{2,0}}, \quad (y)_{\mathbf{1}_{\pm,1}} \otimes \begin{pmatrix} x_2 \\ x_0 \end{pmatrix}_{\mathbf{2}_{2,0}} = \begin{pmatrix} yx_2 \\ yx_0 \end{pmatrix}_{\mathbf{2}_{3,1}}, \\
&(y)_{\mathbf{1}_{\pm,2}} \otimes \begin{pmatrix} x_2 \\ x_0 \end{pmatrix}_{\mathbf{2}_{2,0}} = \begin{pmatrix} yx_0 \\ yx_2 \end{pmatrix}_{\mathbf{2}_{2,0}}, \quad (y)_{\mathbf{1}_{\pm,3}} \otimes \begin{pmatrix} x_2 \\ x_0 \end{pmatrix}_{\mathbf{2}_{2,0}} = \begin{pmatrix} yx_0 \\ yx_2 \end{pmatrix}_{\mathbf{2}_{3,1}}, \\
&(y)_{\mathbf{1}_{\pm,0}} \otimes \begin{pmatrix} x_1 \\ x_0 \end{pmatrix}_{\mathbf{2}_{1,0}} = \begin{pmatrix} yx_1 \\ yx_0 \end{pmatrix}_{\mathbf{2}_{1,0}}, \quad (y)_{\mathbf{1}_{\pm,1}} \otimes \begin{pmatrix} x_1 \\ x_0 \end{pmatrix}_{\mathbf{2}_{1,0}} = \begin{pmatrix} yx_1 \\ yx_0 \end{pmatrix}_{\mathbf{2}_{2,1}}, \\
&(y)_{\mathbf{1}_{\pm,2}} \otimes \begin{pmatrix} x_1 \\ x_0 \end{pmatrix}_{\mathbf{2}_{1,0}} = \begin{pmatrix} yx_1 \\ yx_0 \end{pmatrix}_{\mathbf{2}_{3,2}}, \quad (y)_{\mathbf{1}_{\pm,3}} \otimes \begin{pmatrix} x_1 \\ x_0 \end{pmatrix}_{\mathbf{2}_{1,0}} = \begin{pmatrix} yx_0 \\ yx_1 \end{pmatrix}_{\mathbf{2}_{3,0}}.
\end{aligned} \tag{237}
$$

8.4 $\Sigma(50)$

The $\Sigma(50)$ has fifty elements, $b^k a^m a'^n$ for $k = 0, 1$ and $m,\ n = 0,\ 1,\ 2,\ 3,\ 4$ where a, a' and b satisfy the same conditions as Eq. (193) in the case of $N = 5$. These elements are classified into twenty conjugacy classes,

$$\begin{array}{lcl}
C_1: & \{e\}, & h=1, \\
C_1^{(1)}: & \{aa'\}, & h=5, \\
C_1^{(2)}: & \{a^2a'^2\}, & h=5, \\
C_1^{(3)}: & \{a^3a'^3\}, & h=5, \\
C_1^{(4)}: & \{a^3a'^3\}, & h=5, \\
C_5'^{(0)}: & \{b\ ,\ ba'^2a^3\ ,\ ba'^3a^2\ ,\ ba'^4a\ ,\ ba'a^4\}, & h=2, \\
C_5'^{(1)}: & \{ba'\ ,\ ba\ ,\ ba'^3a^3\ ,\ ba'^4a^2\ ,\ ba'^2a^4\}, & h=10, \\
C_5'^{(2)}: & \{ba'^2\ ,\ ba'a\ ,\ ba^2\ ,\ ba'^4a^3\ ,\ ba'^3a^4\}, & h=10, \\
C_5'^{(3)}: & \{ba'^3\ ,\ ba'^2a\ ,\ ba'a^2\ ,\ ba^3\ ,\ ba'^4a^4\}, & h=10, \\
C_5'^{(4)}: & \{ba'^4\ ,\ ba'^2a^2\ ,\ ba'a^3\ ,\ ba'^3a\ ,\ ba^4\}, & h=10, \\
C_2^{(1,0)}: & \{a\ ,\ a'\}, & h=5, \\
C_2^{(2,0)}: & \{a^2\ ,\ a'^2\}, & h=5, \\
C_2^{(2,1)}: & \{a^2a'\ ,\ aa'^2\}, & h=5, \\
C_2^{(3,0)}: & \{a^3\ ,\ a'^3\}, & h=5, \\
C_2^{(3,1)}: & \{a^3a'\ ,\ aa'^3\}, & h=5, \\
C_2^{(3,2)}: & \{a^3a'^2\ ,\ a^2a'^3\}, & h=5, \\
C_2^{(4,0)}: & \{a^4\ ,\ a'^4\}, & h=5, \\
C_2^{(4,1)}: & \{a^4a'\ ,\ aa'^4\}, & h=5, \\
C_2^{(4,2)}: & \{a^4a'^2\ ,\ a^2a'^4\}, & h=5, \\
C_2^{(4,3)}: & \{a^4a'^3\ ,\ a^3a'^4\}, & h=5,
\end{array} \tag{238}$$

where we have also shown the orders of each element in the conjugacy class by h.

The $\Sigma(50)$ has ten singlets $\mathbf{1}_{\pm,n}$ with $n=0,1,2,3,4$ and ten doublets $\mathbf{2}_{p,q}$ with $(p,q)=(1,0),(2,0),(3,0),(4,0),(2,1),(3,1),(4,1),(3,1),(3,2),(4,3)$. The characters are shown in Tables 18 and 19. Since the tensor products are obtained in the same ways as the cases of lower order, as can been seen from the previous ones, we omit the explicit expressions.

	h	$\chi_{1_{\pm 0}}$	$\chi_{1_{\pm 1}}$	$\chi_{1_{\pm 2}}$	$\chi_{1_{\pm 3}}$	$\chi_{1_{\pm 4}}$
C_1	1	1	1	1	1	1
$C_1^{(1)}$	5	1	ρ^2	ρ^4	ρ	ρ^3
$C_1^{(2)}$	5	1	ρ^4	ρ^3	ρ^2	ρ
$C_1^{(3)}$	5	1	ρ	ρ^2	ρ^3	ρ^4
$C_1^{(4)}$	5	1	ρ^3	ρ	ρ^4	ρ^2
$C_5'^{(0)}$	2	± 1	± 1	± 1	± 1	± 1
$C_5'^{(1)}$	10	± 1	$\pm \rho$	$\pm \rho^2$	$\pm \rho^3$	$\pm \rho^4$
$C_5'^{(2)}$	10	± 1	$\pm \rho^2$	$\pm \rho^4$	$\pm \rho$	$\pm \rho^3$
$C_5'^{(3)}$	10	± 1	$\pm \rho^3$	$\pm \rho$	$\pm \rho^4$	$\pm \rho^2$
$C_5'^{(4)}$	10	± 1	$\pm \rho^4$	$\pm \rho^3$	$\pm \rho^2$	$\pm \rho$
$C_2^{(1,0)}$	5	1	ρ	ρ^2	ρ^3	ρ^3
$C_2^{(2,0)}$	5	1	ρ^2	ρ^4	ρ	ρ^3
$C_2^{(2,1)}$	5	1	ρ^3	ρ	ρ^4	ρ^2
$C_2^{(3,0)}$	5	1	ρ^3	ρ	ρ^4	ρ^2
$C_2^{(3,1)}$	5	1	ρ^4	ρ^3	ρ^2	ρ
$C_2^{(3,2)}$	5	1	1	1	1	1
$C_2^{(4,0)}$	5	1	ρ^4	ρ^3	ρ^2	ρ
$C_2^{(4,1)}$	5	1	1	1	1	1
$C_2^{(4,2)}$	5	1	ρ	ρ^2	ρ^3	ρ^4
$C_2^{(4,3)}$	5	1	ρ^2	ρ^4	ρ	ρ^3

Table 18: Characters of $\Sigma(50)$ representations, where $\rho = e^{2i\pi/5}$

	h	$\chi_{2_{1,0}}$	$\chi_{2_{2,0}}$	$\chi_{2_{2,1}}$	$\chi_{2_{3,0}}$	$\chi_{2_{3,1}}$	$\chi_{2_{3,2}}$	$\chi_{2_{4,0}}$	$\chi_{2_{4,1}}$	$\chi_{2_{4,2}}$	$\chi_{2_{4,3}}$
C_1	1	2	2	2	2	2	2	2	2	2	2
$C_1^{(1)}$	5	2ρ	$2\rho^2$	$2\rho^3$	$2\rho^3$	$2\rho^4$	2	$2\rho^4$	2	2ρ	$2\rho^2$
$C_1^{(2)}$	5	$2\rho^2$	$2\rho^4$	2ρ	2ρ	$2\rho^3$	2	$2\rho^3$	2	$2\rho^2$	$2\rho^4$
$C_1^{(3)}$	5	$2\rho^3$	2ρ	$2\rho^4$	$2\rho^4$	$2\rho^2$	2	$2\rho^2$	2	$2\rho^3$	2ρ
$C_1^{(4)}$	5	$2\rho^4$	$2\rho^3$	$2\rho^2$	$2\rho^2$	2ρ	2	2ρ	2	$2\rho^4$	$2\rho^3$
$C_5'^{(0)}$	2	0	0	0	0	0	0	0	0	0	0
$C_5'^{(1-4)}$	10	0	0	0	0	0	0	0	0	0	0
$C_2^{(1,0)}$	5	$1+\rho$	$1+\rho^2$	$\rho+\rho^2$	$1+\rho^3$	$\rho+\rho^3$	$\rho^2+\rho^3$	$1+\rho^4$	$\rho+\rho^4$	$\rho^2+\rho^4$	$\rho^3+\rho^4$
$C_2^{(2,0)}$	5	$1+\rho^2$	$1+\rho^4$	$\rho^2+\rho^4$	$1+\rho$	$\rho+\rho^2$	$\rho+\rho^4$	$1+\rho^3$	$\rho^2+\rho^3$	$\rho^3+\rho^4$	$\rho+\rho^3$
$C_2^{(2,1)}$	5	$\rho+\rho^2$	$\rho^2+\rho^4$	$1+\rho^4$	$\rho+\rho^3$	$1+\rho^2$	$\rho^2+\rho^3$	$\rho^3+\rho^4$	$\rho+\rho^4$	$1+\rho^3$	$1+\rho$
$C_2^{(3,0)}$	5	$1+\rho^3$	$1+\rho$	$\rho+\rho^3$	$1+\rho^4$	$\rho^3+\rho^4$	$\rho+\rho^4$	$1+\rho^2$	$\rho^2+\rho^3$	$\rho+\rho^2$	$\rho^2+\rho^4$
$C_2^{(3,1)}$	5	$\rho+\rho^3$	$\rho+\rho^2$	$1+\rho^2$	$\rho^3+\rho^4$	$1+\rho$	$\rho+\rho^4$	$\rho^2+\rho^4$	$\rho^2+\rho^3$	$1+\rho^4$	$1+\rho^3$
$C_2^{(3,2)}$	5	$\rho^2+\rho^3$	$\rho+\rho^4$	$\rho^2+\rho^3$	$\rho+\rho^4$	$\rho+\rho^4$	$\rho^2+\rho^3$	$\rho^2+\rho^3$	$\rho+\rho^4$	$\rho+\rho^4$	$\rho^2+\rho^3$
$C_2^{(4,0)}$	5	$1+\rho^4$	$1+\rho^3$	$\rho^3+\rho^4$	$1+\rho^2$	$\rho^2+\rho^4$	$\rho^2+\rho^3$	$1+\rho$	$\rho+\rho^4$	$\rho+\rho^3$	$\rho+\rho^2$
$C_2^{(4,1)}$	5	$\rho+\rho^4$	$\rho^2+\rho^3$	$\rho+\rho^4$	$\rho^2+\rho^3$	$\rho^2+\rho^3$	$\rho+\rho^4$	$\rho+\rho^4$	$\rho^2+\rho^3$	$\rho^2+\rho^3$	$\rho+\rho^4$
$C_2^{(4,2)}$	5	$\rho^2+\rho^4$	$\rho^3+\rho^4$	$1+\rho^3$	$\rho+\rho^2$	$1+\rho^4$	$1+\rho$	$\rho+\rho^3$	$\rho^2+\rho^3$	$1+\rho$	$1+\rho^2$
$C_2^{(4,3)}$	5	$\rho^3+\rho^4$	$\rho+\rho^3$	$1+\rho$	$\rho^2+\rho^4$	$1+\rho^3$	$\rho^2+\rho^3$	$\rho+\rho^2$	$\rho+\rho^4$	$1+\rho^2$	$1+\rho^4$

Table 19: Characters of $\Sigma(50)$ representations, where $\rho = e^{2i\pi/5}$

9 $\Delta(3N^2)$

9.1 Generic aspects

The discrete group $\Delta(3N^2)$ is isomorphic to $(Z_N \times Z'_N) \rtimes Z_3$. (See also Ref. [189].) We denote the generators of Z_N and Z'_N by a and a', respectively, and the Z_3 generator is written by b. They satisfy

$$a^N = a'^N = b^3 = e, \qquad aa' = a'a,$$
$$bab^{-1} = a^{-1}(a')^{-1}, \qquad ba'b^{-1} = a. \tag{239}$$

Using them, all of $\Delta(3N^2)$ elements are written as

$$g = b^k a^m a'^n, \tag{240}$$

for $k = 0, 1, 2$ and $m, n = 0, 1, 2, \cdots, N-1$.

The generators, a, a' and b, are represented, e.g. as

$$b = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, \quad a = \begin{pmatrix} \rho & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \rho^{-1} \end{pmatrix}, \quad a' = \begin{pmatrix} \rho^{-1} & 0 & 0 \\ 0 & \rho & 0 \\ 0 & 0 & 1 \end{pmatrix}, \tag{241}$$

where $\rho = e^{2\pi i/N}$. Then, all elements of $\Delta(3N^2)$ are represented as

$$\begin{pmatrix} \rho^m & 0 & 0 \\ 0 & p^n & 0 \\ 0 & 0 & \rho^{-m-n} \end{pmatrix}, \quad \begin{pmatrix} 0 & \rho^m & 0 \\ 0 & 0 & \rho^n \\ \rho^{-m-n} & 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 & \rho^m \\ \rho^n & 0 & 0 \\ 0 & \rho^{-m-n} & 0 \end{pmatrix}, \tag{242}$$

for $m, n = 0, 1, 2, \cdots, N-1$.

• **Conjugacy classes**

Now, let us study the conjugacy classes. It is found that

$$ba^\ell a'^m b^{-1} = a^{-\ell+m} a'^{-\ell}, \qquad b^2 a^\ell a'^m b^{-2} = a^{-m} a'^{\ell-m}. \tag{243}$$

Thus, these elements, $a^\ell a'^m$, $a^{-\ell+m}a'^{-\ell}$, $a^{-m}a'^{\ell-m}$, must belong to the same conjugacy class. These are independent elements of $\Delta(3N^2)$ unless $N/3 =$ integer and $3\ell = \ell+m = 0$ (mod N). On the other hand, if $N/3 =$ integer and $3\ell = \ell + m = 0$ (mod N), the above elements are the same, i.e. $a^\ell a'^{-\ell}$. As a result, the elements $a^\ell a'^m$ are classified into the following conjugacy classes,

$$C_3^{(\ell,m)} = \{a^\ell a'^m,\ a^{-\ell+m}a'^{-\ell},\ a^{-m}a'^{\ell-m}\}, \tag{244}$$

for $N/3 \neq$ integer,

$$C_1^\ell = \{a^\ell a'^{-\ell}\}, \qquad \ell = \frac{N}{3}, \frac{2N}{3},$$
$$C_3^{(\ell,m)} = \{a^\ell a'^m,\ a^{-\ell+m}a'^{-\ell},\ a^{-m}a'^{\ell-m}\}, \quad (\ell, m) \neq \left(\frac{N}{3}, \frac{2N}{3}\right), \left(\frac{2N}{3}, \frac{N}{3}\right), \tag{245}$$

for $N/3 =$ integer.

Similarly, we can obtain conjugacy classes including $ba^\ell a'^m$. Let us consider the conjugates of the simplest element b among $ba^\ell a'^m$. It is found that

$$a^p a'^q (b) a^{-p} a'^{-q} = ba^{-p-q} a'^{p-2q} = ba^{-n+3q} a'^n, \tag{246}$$

where we have written for convenience by using $n \equiv p - 2q$. We also obtain

$$b(ba^{-n+3q}a'^n)b^{-1} = ba^{2n-3q}a'^{n-3q}, \tag{247}$$
$$b^2(ba^{-n+3q}a'^n)b^{-2} = ba^{-n}a'^{-2n+3q}. \tag{248}$$

The important property is that q appears only in the form of $3q$. Thus, if $N/3 \neq$ integer, the element of b is conjugate to all of $ba^\ell a'^m$. That is, all of them belong to the same conjugacy class $C^1_{N^2}$. Similarly, all of $b^2 a^\ell a'^m$ belong to the same conjugacy class $C^2_{N^2}$ for $N/3 \neq$ integer.

On the other hand, if $N/3 =$ integer, the situation is different. Among the above elements conjugate to b, there does not appear ba. Its conjugates are also obtained as

$$a^p a'^q (ba) a^{-p} a'^{-q} = ba^{1-p-q} a'^{p-2q} = ba^{1-n+3q} a'^n, \tag{249}$$
$$b(ba^{1-n+3q}a'^n)b^{-1} = ba^{-1+2n-3q}a'^{-1+n-3q}, \tag{250}$$
$$b^2(ba^{1-n+3q}a'^n)b^{-2} = ba^{-n}a'^{1-2n+3q}. \tag{251}$$

It is found that these elements conjugate to ba as well as conjugates of b do not include ba^2 when $N/3 =$ integer. As a result, it is found that for $N/3 =$ integer, the elements $ba^\ell a'^m$ are classified into three conjugacy classes, $C^{(\ell)}_{N^2/3}$ for $\ell = 0, 1, 2$, i.e.,

$$C^{(\ell)}_{N^2/3} = \{ba^{\ell-n-3m}a'^n | m = 0, 1, ..., \frac{N-3}{3};\ n = 0, ..., N-1\}. \tag{252}$$

Similarly, the $b^2 a^\ell a'^m$ are classified into three conjugacy classes, $C^{(\ell)}_{N^2/3}$ for $\ell = 0, 1, 2$, i.e.,

$$C^{(\ell)}_{N^2/3} = \{b^2a^{\ell-n-3m}a'^n | m = 0, 1, ..., \frac{N-3}{3};\ n = 0, ..., N-1\}. \tag{253}$$

Here, we summarize the conjugacy classes of $\Delta(3N^2)$. For $N/3 \neq$ integer, the $\Delta(3N^2)$ has the following conjugacy classes,

$$\begin{array}{rccc} C_1: & \{e\}, & h = 1, \\ C_3^{(\ell,m)}: & \{a^\ell a'^m, a^{-\ell+m}a'^{-\ell}, a^{-\sigma}a'^{\ell-m}\}, & h = N/\gcd(N,\ell,m), \\ C^1_{N^2}: & \{ba^\ell a'^m | \ell, m = 0, 1, \cdots, N-1\}, & h = N/\gcd(N,3,\ell,m), \\ C^2_{N^2}: & \{b^2a^\ell a'^m | \ell, m = 0, 1, \cdots, N-1\}, & h = N/\gcd(N,3,\ell,m). \end{array} \tag{254}$$

The number of the conjugacy classes $C_3^{(\ell,m)}$ is equal to $(N^2-1)/3$. Then, the total number of conjugacy classes is equal to $3 + (N^2-1)/3$. The relations (12) and (13) for $\Delta(3n^2)$ with $N/3 \neq$ integer lead to

$$m_1 + 2^2 m_2 + 3^2 m_3 + \cdots = 3N^2, \tag{255}$$
$$m_1 + m_2 + m_3 + \cdots = 3 + (N^2-1)/3. \tag{256}$$

The solution is found as $(m_1, m_3) = (3, (N^2-1)/3)$. That is, there are three singlets and $(N^2-1)/3$ triplets.

On the other hand, for $N/3 =$ integer, the $\Delta(3N^2)$ has the following conjugacy classes,

$$\begin{array}{cc} C_1: & \{e\}, \\ C_1^{(k)}: & \{a^k a'^{-k}\}, \quad k = \frac{N}{3}, \frac{2N}{3}, \\ C_3^{(\ell,m)}: & \{a^\ell a'^m, a^{-\ell+m}a'^{-\ell}, a^{-m}a'^{\ell-m}\}, \quad (\ell,m) \neq \left(\frac{N}{3}, \frac{2N}{3}\right), \left(\frac{2N}{3}, \frac{N}{3}\right), \\ C_{N^2/3}^{(1,p)}: & \{ba^{p-n-3m}a'^n | m = 0,1,\cdots,\frac{N-3}{3}, n = 0,1,\cdots,N-1\}, \quad p = 0,1,2, \\ C_{N^2/3}^{(2,p)}: & \{b^2a^{p-n-3m}a'^n | m = 0,1,\cdots,\frac{N-3}{3}, n = 0,1,\cdots,N-1\}, \quad p = 0,1,2. \end{array} \tag{257}$$

The orders of each element in the conjugacy classes, i.e. $g^h = e$ are obtained as follows,

$$\begin{array}{cc} C_1: & h = 1, \\ C_1^{(k)}: & h = N/\gcd(N, N/3, 2N/3), \\ C_3^{(\ell,m)}: & h = N/\gcd(N,\ell,m), \\ C_{N^2/3}^{(1,p)}: & h = N/\gcd(N,3,p-n-3m,n), \\ C_{N^2/3}^{(2,p)}: & h = N/\gcd(N,3,p-n-3m,n). \end{array} \tag{258}$$

The numbers of the conjugacy classes $C_1^{(k)}$, $C_3^{(\ell,m)}$, $C_{N^2/3}^{(1,p)}$ and $C_{N^2/3}^{(2,p)}$, are equal to 2, $(N^2-3)/3$, 3 and 3, respectively. The total number of conjugacy classes is equal to $9 + (N^2-3)/3$. The relations (12) and (13) for $\Delta(3N^2)$ with $N/3 =$ integer lead to

$$m_1 + 2^2 m_2 + 3^2 m_3 + \cdots = 3N^2, \tag{259}$$

$$m_1 + m_2 + m_3 + \cdots = 9 + (N^2-3)/3. \tag{260}$$

The solution is found as $(m_1, m_3) = (9, (N^2-3)/3)$. That is, there are nine singlets and $(N^2-3)/3$ triplets.

• **Characters and representations**

Now, we study characters. We start with $N/3 \neq$ integer. In this case, there are 3 singlets. Because $b^3 = e$, characters of three singlets have three possible values $\chi_{1k}(b) = \omega^k$ with $k = 0, 1, 2$ and they correspond to three singlets, $\mathbf{1}_k$. Note that $\chi_{1k}(a) = \chi_{1k}(a') = 1$, because $\chi_{1k}(b) = \chi_{1k}(ba) = \chi_{1k}(ba')$. These characters are shown in Table 20.

Next, let us consider triplets for $N/3 \neq$ integer. Indeed, the matrices (241) correspond to one of triplet representations. Similarly, we can obtain (3×3) matrix representations for generic triplets, e.g. by replacing

$$a \to a^\ell a'^m, \qquad a' \to b^2 a b^{-2} = a^{-m} a'^{\ell-m}. \tag{261}$$

However, note that the following two types of replacing

$$\begin{aligned} (a, a') &\to (a^{-\ell+m}a'^{-\ell}, a^\ell a'^m), \\ (a, a') &\to (a^{-m}a'^{\ell-m}, a^{-\ell+m}a'^{-\ell}), \end{aligned} \tag{262}$$

	h	χ_{1_0}	χ_{1_1}	χ_{1_2}	$\chi_{3_{[0][1]}}$	$\chi_{3_{[0][2]}}$	$\cdots$	$\chi_{3_{[n-1][n-1]}}$
C_1	1	1	1	1	3	3		3
$C_3^{(\ell,m)}$	$\frac{N}{\gcd(N,\ell,m)}$	1	1	1	$\rho^{\ell-m}$ $+\rho^{m}+\rho^{-\ell}$	$\rho^{2\ell-2m}$ $+\rho^{2m}+\rho^{-2\ell}$		$\rho^{n-1}(\rho^{\ell-2m}$ $+\rho^{\ell+m}+\rho^{-2\ell+m})$
$C_{N^2}^1$	$\frac{N}{\gcd(N,3,\ell,m)}$	1	ω	ω^2	0	0		0
$C_{N^2}^2$	$\frac{N}{\gcd(N,3,\ell,m)}$	1	ω^2	ω	0	0		0

Table 20: Characters of $\Delta(3N^2)$ for $N/3 \neq$ integer

also lead to the representation equivalent to the above (261), because the three elements, $a^\ell a'^m$, $a^{-\ell+m}a'^{-\ell}$ and $a^{-m}a'^{\ell-m}$, belong to the same conjugacy class, $C_3^{(\ell,m)}$. Thus, the $\Delta(3N^2)$ group with $N/3 \neq$ integer is represented as

$$b = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, \quad a = \begin{pmatrix} \rho^\ell & 0 & 0 \\ 0 & \rho^k & 0 \\ 0 & 0 & \rho^{-k-\ell} \end{pmatrix}, \quad a' = \begin{pmatrix} \rho^{-k-\ell} & 0 & 0 \\ 0 & \rho^\ell & 0 \\ 0 & 0 & \rho^k \end{pmatrix}, \tag{263}$$

on the triplet $\mathbf{3}_{[k][\ell]}$, where $[k][\ell]$ denotes [1]

$$[k][\ell] = (k,\ell), \quad (-k-\ell,k) \text{ or } (\ell,-k-\ell). \tag{264}$$

We also denote the vector of $\mathbf{3}_{[k][\ell]}$ as

$$\mathbf{3}_{[k][\ell]} = \begin{pmatrix} x_{\ell,-k-\ell} \\ x_{k,\ell} \\ x_{-k-\ell,k} \end{pmatrix}, \tag{265}$$

for $k,\ell = 0,1,\cdots,N-1$, where k and ℓ correspond to Z_N and Z'_N charges, respectively. When $(k,\ell) = (0,0)$, the matrices a and a' are proportional to the identity matrix. Thus, we exclude the case with $(k,\ell) = (0,0)$. The characters are shown in Table 20.

Similarly, we study the characters and representations for $N/3 =$ integer. In this case, there are nine singlets. Their characters must satisfy $\chi_\alpha(b) = \omega^k$ $(k = 0,1,2)$ similarly to the above case. In addition, it is found that $\chi_\alpha(a) = \chi_\alpha(a') = \omega^\ell$ $(\ell = 0,1,2)$. Thus, nine singlets can be specified by combinations of $\chi_\alpha(b)$ and $\chi_\alpha(a)$, i.e. $\mathbf{1}_{k,\ell}$ $(k,\ell = 0,1,2)$ with $\chi_\alpha(b) = \omega^k$ and $\chi_\alpha(a) = \chi_\alpha(a') = \omega^\ell$. These characters are shown in Table 21.

The triplet representations are also given similarly to the case with $N/3 \neq$ integer. That is, the $\Delta(3N^2)$ group with $N/3 =$ integer is represented as

$$b = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, \quad a = \begin{pmatrix} \rho^\ell & 0 & 0 \\ 0 & \rho^k & 0 \\ 0 & 0 & \rho^{-k-\ell} \end{pmatrix}, \quad a' = \begin{pmatrix} \rho^{-k-\ell} & 0 & 0 \\ 0 & \rho^\ell & 0 \\ 0 & 0 & \rho^k \end{pmatrix}, \tag{266}$$

on the triplet $\mathbf{3}_{[k][\ell]}$. However, note that when $(k,\ell) = (0,0),(N/3,N/3),(2N/3,2N/3)$ the matrices, a and a' are trivial. Thus, we exclude such values of (k,ℓ). These characters are shown in Table 21.

[1]The notation $[k][\ell]$ corresponds to $\widetilde{(k,\ell)}$ in Ref. [189].

	h	$\chi_{1_{(r,s)}}$	$\chi_{3_{[0][1]}}$	$\chi_{3_{[0][2]}}$	$\cdots$	$\chi_{3_{[n-1][n-1]}}$
$1C_1$	1	1	3	3		3
$1C_1^{(k)}$	$\frac{N}{\gcd(N,N/3,2N/3)}$	1	$\rho^{2k}+2\rho^{-k}$	$\rho^{4k}+2\rho^{-2k}$		$1+\rho^{3(n-1)\rho}+\rho^{-3(n-1)\rho}$
$3C_1^{(\ell,m)}$	$\frac{N}{\gcd(N,\ell,m)}$	$\omega^{s(\ell+m)}$	$\rho^{\ell-m}$ $+\rho^{m}+\rho^{-\ell}$	$\rho^{2\ell-2m}$ $+\rho^{2m}+\rho^{-2\ell}$		$\rho^{n-1}(\rho^{\ell-2m}$ $+\rho^{\ell+m}+\rho^{-2\ell+m})$
$C_{N^2/3}^{(1,p)}$	$\frac{N}{\gcd(N,3,p-n-3m,n)}$	ω^{r+sp}	0	0		0
$C_{N^2/3}^{(2,p)}$	$\frac{N}{\gcd(N,3,p-n-3m,n)}$	ω^{2r+sp}	0	0		0

Table 21: Characters of $\Delta(3N^2)$ for $N/3=$ integer

• **Tensor products**

Now, we study tensor products. First, we consider the $\Delta(3N^2)$ with $N/3\neq$ integer. Because of their $Z_N\times Z'_N$ charges, tensor products of triplets **3** can be obtained as

$$\begin{pmatrix} x_{\ell,-k-\ell} \\ x_{k,\ell} \\ x_{-k-\ell,k} \end{pmatrix}_{\mathbf{3}_{[k][\ell]}} \otimes \begin{pmatrix} y_{\ell',-k'-\ell'} \\ y_{k',\ell'} \\ y_{-k'-\ell',k'} \end{pmatrix}_{\mathbf{3}_{[k'][\ell']}} = \begin{pmatrix} x_{\ell,-k-\ell}y_{\ell',-k'-\ell'} \\ x_{k,\ell}y_{k',\ell'} \\ x_{-k-\ell,k}y_{-k'-\ell',k'} \end{pmatrix}_{\mathbf{3}_{([k+k'][\ell+\ell']}}$$
$$\oplus \begin{pmatrix} x_{-k-\ell,k}y_{\ell',-k'-\ell'} \\ x_{\ell,-k-\ell}y_{k',\ell'} \\ x_{k,\ell}y_{-k'-\ell',k'} \end{pmatrix}_{\mathbf{3}_{[\ell+k'][-k-\ell+\ell']}} \oplus \begin{pmatrix} x_{\ell,-k-\ell}y_{-k'-\ell',k'} \\ x_{k,\ell}y_{\ell',-k'-\ell'} \\ x_{-k-\ell,k}y_{k',\ell'} \end{pmatrix}_{\mathbf{3}_{[k+\ell'][\ell-k'-\ell']}} \qquad (267)$$

for $-(k,\ell)\neq[k'][\ell']$,

$$\begin{pmatrix} x_{\ell,-k-\ell} \\ x_{k,\ell} \\ x_{-k-\ell,k} \end{pmatrix}_{\mathbf{3}_{[k][\ell]}} \otimes \begin{pmatrix} y_{-\ell,k+\ell} \\ y_{-k,-\ell} \\ y_{k+\ell,-k} \end{pmatrix}_{\mathbf{3}_{-[k][\ell]}} = (x_{\ell,-k-\ell}y_{-\ell,k+\ell}+x_{k,\ell}y_{-k,-\ell}+x_{-k-\ell,k}y_{k+\ell,-k})_{\mathbf{1}_0}$$
$$\oplus\ (x_{\ell,-k-\ell}y_{-\ell,k+\ell}+\omega^2x_{k,\ell}y_{-k,-\ell}+\omega x_{-k-\ell,k}y_{k+\ell,-k})_{\mathbf{1}_1}$$
$$\oplus\ (x_{\ell,-k-\ell}y_{-\ell,k+\ell}+\omega x_{k,\ell}y_{-k,-\ell}+\omega^2x_{-k-\ell,k}y_{k+\ell,-k})_{\mathbf{1}_2}$$
$$\oplus \begin{pmatrix} x_{-k-\ell,k}y_{-\ell,k+\ell} \\ x_{\ell,-k-\ell}y_{-k,-\ell} \\ x_{k,\ell}y_{k+\ell,-k} \end{pmatrix}_{\mathbf{3}_{[-k+\ell][-k-2\ell]}} \oplus \begin{pmatrix} x_{\ell,-k-\ell}y_{k+\ell,-k} \\ x_{k,\ell}y_{-\ell,k+\ell} \\ x_{-k-\ell,k}y_{-k,-\ell} \end{pmatrix}_{\mathbf{3}_{[k-\ell][k+2\ell]}} . \qquad (268)$$

A product of $\mathbf{3}_{[k][\ell]}$ and $\mathbf{1}_r$ is obtained as

$$\begin{pmatrix} x_{(\ell,-k-\ell)} \\ x_{(k,\ell)} \\ x_{(-k-\ell,k)} \end{pmatrix}_{[k][\ell]} \otimes (z_r)_{\mathbf{1}_r} = \begin{pmatrix} x_{(\ell,-k-\ell)}z_r \\ \omega^r x_{(k,\ell)}z_r \\ \omega^{2r}x_{(-k-\ell,k)}z_r \end{pmatrix}_{[k][\ell]} . \qquad (269)$$

The tensor products of singlets $\mathbf{1}_k$ and $\mathbf{1}_{k'}$ are obtained simply as

$$\mathbf{1}_k\otimes\mathbf{1}_{k'}=\mathbf{1}_{k+k'}. \qquad (270)$$

Next, we study tensor products of $N/3 =$ integer. We consider the tensor products of two triplets,

$$\begin{pmatrix} x_{\ell,-k-\ell} \\ x_{k,\ell} \\ x_{-k-\ell,k} \end{pmatrix}_{\mathbf{3}_{[k][\ell]}} \quad \text{and} \quad \begin{pmatrix} y_{\ell',-k'-\ell'} \\ y_{k',\ell'} \\ y_{-k'-\ell',k'} \end{pmatrix}_{\mathbf{3}_{[k'][\ell']}} . \tag{271}$$

Unless $(k', \ell') = -[k + mN/3][\ell + mN/3]$ for $m = 0, 1, 2$, their tensor products are the same as (267). Thus, we do not repeat them. For $(k', \ell') = -[-k + mN/3][-\ell + mN/3]$ ($m = 0, 1$ or 2), tensor products of the above triplets are obtained as

$$\begin{aligned}
&\begin{pmatrix} x_{\ell,-k-\ell} \\ x_{k,\ell} \\ x_{-k-\ell,k} \end{pmatrix}_{\mathbf{3}_{[k][\ell]}} \otimes \begin{pmatrix} y_{-\ell+mN/3,k+\ell-2mN/3} \\ y_{-k+mN/3,-\ell+mN/3} \\ y_{k+\ell-2mN/3,-k+mN/3} \end{pmatrix}_{\mathbf{3}_{[-k+mN/3][-\ell+mN/3]}} \\
&= \left(x_{\ell,-k-\ell}y_{-\ell,k+\ell-2mN/3} + x_{k,\ell}y_{-k+mN/3,-\ell+mN/3} + x_{-k-\ell,k}y_{k+\ell-2mN/3,-k+mN/3}\right)_{\mathbf{1}_{0,m}} \\
&\oplus \left(x_{\ell,-k-\ell}y_{-\ell+mN/3,k+\ell-2mN/3} + \omega^2 x_{k,\ell}y_{-k+mN/3,-\ell+mN/3} + \omega x_{-k-\ell,k}y_{k+\ell-2mN/3,-k+mN/3}\right)_{\mathbf{1}_{1,m}} \\
&\oplus \left(x_{\ell,-k-\ell}y_{-\ell+mN/3,k+\ell-2mN/3} + \omega x_{k,\ell}y_{-k+mN/3,-\ell+mN/3} + \omega^2 x_{-k-\ell,k}y_{k+\ell-2mN/3,-k+mN/3}\right)_{\mathbf{1}_{2,m}} \\
&\oplus \begin{pmatrix} x_{-k-\ell,k}y_{-\ell+mN/3,k+\ell-2mN/3} \\ x_{\ell,-k-\ell}y_{-k+mN/3,-\ell+mN/3} \\ x_{k,\ell}\phi'_{k+\ell-2mN/3,-k+mN/3} \end{pmatrix}_{\mathbf{3}_{[-k+\ell+mN/3][-k-2\ell+mN/3]}} \\
&\oplus \begin{pmatrix} x_{\ell,-k-\ell}y_{k+\ell-2mN/3,-k+mN/3} \\ x_{k,\ell}y_{-\ell+mN/3,k+\ell-2mN/3} \\ x_{-k-\ell,k}y_{-k+mN/3,-\ell+mN/3} \end{pmatrix}_{\mathbf{3}_{[k-\ell+mN/3][k+2\ell-2mN/3]}} .
\end{aligned} \tag{272}$$

A product of $\mathbf{3}_{[k][\ell]}$ and $\mathbf{1}_{r,s}$ is

$$\begin{pmatrix} x_{(\ell,-k-\ell)} \\ x_{(k,\ell)} \\ x_{(-k-\ell,k)} \end{pmatrix}_{\mathbf{3}_{[k][\ell]}} \otimes (z_{r,s})_{\mathbf{1}_{r,s}} = \begin{pmatrix} x_{(\ell,-k-\ell)}z_{r,s} \\ \omega^r x_{(k,\ell)}z_{r,s} \\ \omega^{2r} x_{(-k-\ell,k)}z_{r,s} \end{pmatrix}_{\mathbf{3}_{[k+sN/3][\ell+sN/3]}} . \tag{273}$$

The tensor products of singlets $\mathbf{1}_{k,\ell}$ and $\mathbf{1}_{k',\ell'}$ are obtained simply as

$$\mathbf{1}_{k,\ell} \otimes \mathbf{1}_{k',\ell'} = \mathbf{1}_{k+k',\ell+\ell'} . \tag{274}$$

9.2 $\Delta(27)$

The $\Delta(3)$ is nothing but the Z_3 group and $\Delta(12)$ is isomorphic to A_4. Thus, the simple and non-trivial example is $\Delta(27)$.

	h	$\chi_{1_{(r,s)}}$	$\chi_{3_{[0,1]}}$	$\chi_{3_{[0,2]}}$
$1C_1$	1	1	3	3
$1C_1^{(1)}$	1	1	$3\omega^2$	3ω
$1C_1^{(2)}$	1	1	3ω	$3\omega^2$
$3C_1^{(0,1)}$	3	ω^s	0	0
$3C_1^{(0,2)}$	3	ω^{2s}	0	0
$C_3^{(1,p)}$	3	ω^{r+sp}	0	0
$C_3^{(2,p)}$	3	ω^{2r+sp}	0	0

Table 22: Characters of $\Delta(27)$

The conjugacy classes of $\Delta(27)$ are obtained as

$$\begin{array}{rcl} C_1: & \{e\}, & h=1, \\ C_1^{(1)}: & \{a, a'^2\}, & h=3, \\ C_1^{(2)}: & \{a^2, a'\}, & h=3, \\ C_3^{(0,1)}: & \{a', a, a^2a'^2\}, & h=3, \\ C_3^{(0,2)}: & \{a'^2, a^2, aa'\}, & h=3, \\ C_3^{(1,p)}: & \{ba^p, ba^{p-1}a', ba^{p-2}a'^2\}, & h=3, \\ C_3^{(2,p)}: & \{ba^p, ba^{p-1}a', ba^{p-2}a'^2\}, & h=3. \end{array} \tag{275}$$

The $\Delta(27)$ has nine singlets $\mathbf{1}_{r,s}$ $(r,s=0,1,2)$ and two triplets, $\mathbf{3}_{[0][1]}$ and $\mathbf{3}_{[0][2]}$. The characters are shown in Table 22.

Tensor products between triplets are obtained as

$$\begin{pmatrix} x_{1,-1} \\ x_{0,1} \\ x_{-1,0} \end{pmatrix}_{\mathbf{3}_{[0][1]}} \otimes \begin{pmatrix} y_{1,-1} \\ y_{0,1} \\ y_{-1,0} \end{pmatrix}_{\mathbf{3}_{[0][1]}} = \begin{pmatrix} x_{1,-1}y_{1,-1} \\ x_{0,1}y_{0,1} \\ x_{-1,0}y_{-1,0} \end{pmatrix}_{\mathbf{3}_{[0][2]}} \oplus \begin{pmatrix} x_{-1,0}y_{1,-1} \\ x_{1,-1}y_{0,1} \\ x_{0,1}y_{-1,0} \end{pmatrix}_{\mathbf{3}_{[0][2]}} \oplus \begin{pmatrix} x_{1,-1}y_{-1,0} \\ x_{0,1}y_{1,-1} \\ x_{-1,0}y_{0,1} \end{pmatrix}_{\mathbf{3}_{[0][2]}}, \tag{276}$$

$$\begin{pmatrix} x_{2,-2} \\ x_{0,2} \\ x_{-2,0} \end{pmatrix}_{\mathbf{3}_{[0][2]}} \otimes \begin{pmatrix} y_{2,-2} \\ y_{0,2} \\ y_{-2,0} \end{pmatrix}_{\mathbf{3}_{[0][2]}} = \begin{pmatrix} x_{2,-2}y_{2,-2} \\ x_{0,2}y_{0,2} \\ x_{-2,0}y_{-2,0} \end{pmatrix}_{\mathbf{3}_{[0][1]}} \oplus \begin{pmatrix} x_{-2,0}y_{2,-2} \\ x_{2,-2}y_{0,2} \\ x_{0,2}y_{-2,0} \end{pmatrix}_{\mathbf{3}_{[0][1]}} \oplus \begin{pmatrix} x_{2,-2}y_{-2,0} \\ x_{0,2}y_{2,-2} \\ x_{-2,0}y_{0,2} \end{pmatrix}_{\mathbf{3}_{[0][1]}}, \tag{277}$$

$$\begin{pmatrix} x_{1,-1} \\ x_{0,1} \\ x_{-1,0} \end{pmatrix}_{\mathbf{3}_{[0][1]}} \otimes \begin{pmatrix} y_{-1,1} \\ y_{0,-1} \\ y_{1,0} \end{pmatrix}_{\mathbf{3}_{[0][2]}} = \sum_r (x_{1,-1}y_{-1,1} + \omega^{2r} x_{0,1}y_{0,-1} + \omega^r x_{-1,0}y_{1,0})_{\mathbf{1}_{(r,0)}}$$
$$\oplus \sum_r (x_{1,-1}y_{0,-1} + \omega^{2r} x_{0,1}y_{1,0} + \omega^r x_{-1,0}y_{-1,1})_{\mathbf{1}_{(r,1)}}$$
$$\oplus \sum_r (x_{1,-1}y_{1,0} + \omega^{2r} x_{0,1}y_{-1,1} + \omega^r x_{-1,0}y_{0,-1})_{\mathbf{1}_{(r,2)}}. \tag{278}$$

The tensor products between singlets and triplets are obtained as

$$\begin{pmatrix} x_{(1,-1)} \\ x_{(0,1)} \\ x_{(-1,0)} \end{pmatrix}_{\mathbf{3}_{[0][1]}} \otimes (z_{r,s})_{1_{r,s}} = \begin{pmatrix} x_{(1,-1)}z_{r,s} \\ \omega^r x_{(0,1)}z_{r,s} \\ \omega^{2r} x_{(-1,0)}z_{r,s} \end{pmatrix}_{\mathbf{3}_{[s][1+s]}},$$
$$\begin{pmatrix} x_{(2,-2)} \\ x_{(0,2)} \\ x_{(-2,0)} \end{pmatrix}_{\mathbf{3}_{[0][2]}} \otimes (z_{r,s})_{1_{r,s}} = \begin{pmatrix} x_{(2,-2)}z_{r,s} \\ \omega^r x_{(0,2)}z_{r,s} \\ \omega^{2r} x_{(-2,0)}z_{r,s} \end{pmatrix}_{\mathbf{3}_{[s][2+s]}}. \tag{279}$$

The tensor products of singlets are obtained as Eq. (274).

	n	h	$\chi_{\mathbf{1_0}}$	$\chi_{\mathbf{1_1}}$	$\chi_{\mathbf{1_2}}$	$\chi_{\mathbf{3}}$	$\chi_{\bar{\mathbf{3}}}$
$C_1^{(0)}$	1	1	1	1	1	3	3
$C_7^{(1)}$	7	3	1	ω	ω^2	0	0
$C_7^{(2)}$	7	3	1	ω^2	ω	0	0
C_3	3	7	1	1	1	ξ	$\bar{\xi}$
$C_{\bar{3}}$	3	7	1	1	1	$\bar{\xi}$	ξ

Table 23: Characters of T_7

10 T_7

It is useful to construct a discrete group as a subgroup of known groups. Through such a procedure, we can obtain group-theoretical aspects such as representations of the subgroup from those of larger groups. As such an example, here we study T_7, which is isomorphic to $Z_7 \rtimes Z_3$ and a subgroup of $\Delta(3N^2)$ with $N = 7$. The discrete group T_7[195] is known as the minimal non-Abelian discrete group with respect to having a complex triplet.

We denote the generators of Z_7 by a and Z_3 generator is written by b. They satisfy

$$a^7 = 1, \quad ab = ba^4. \tag{280}$$

Using them, all of T_7 elements are written as

$$g = b^m a^n, \tag{281}$$

with $m = 0, 1, 2$ and $n = 0, \cdots, 6$.

The generators, a and b, are represented e.g. as

$$b = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, \quad a = \begin{pmatrix} \rho & 0 & 0 \\ 0 & \rho^2 & 0 \\ 0 & 0 & \rho^4 \end{pmatrix}, \tag{282}$$

where $\rho = e^{2i\pi/7}$. These elements are classified into five conjugacy classes,

$$\begin{array}{rcl} C_1: & \{e\}, & h=1, \\ C_7^{(1)}: & \{b \ , \ ba \ , \ ba^2 \ , \ ba^3 \ , \ ba^4 \ , \ ba^5 \ , \ ba^6\}, & h=3, \\ C_7^{(2)}: & \{b^2 \ , \ b^2a \ , \ b^2a^2 \ , \ b^2a^3 \ , \ b^2a^4 \ , \ b^2a^5 \ , \ b^2a^6\}, & h=3, \\ C_3: & \{a \ , \ a^2 \ , \ a^4\}, & h=7, \\ C_{\bar{3}}: & \{a^3 \ , \ a^5 \ , \ a^6\}, & h=7. \end{array} \tag{283}$$

The T_7 group has three singlets $\mathbf{1}_k$ with $k = 0, 1, 2$ and two triplets $\mathbf{3}$ and $\bar{\mathbf{3}}$. The characters are shown in Table 23, where $\xi = \frac{-1+i\sqrt{7}}{2}$.

Using the order of ρ in a, we define the triplet $\mathbf{3}(\bar{\mathbf{3}})$ as

$$\mathbf{3} \equiv \begin{pmatrix} x_1 \\ x_2 \\ x_4 \end{pmatrix}, \quad \bar{\mathbf{3}} \equiv \begin{pmatrix} x_{-1} \\ x_{-2} \\ x_{-4} \end{pmatrix} = \begin{pmatrix} x_6 \\ x_5 \\ x_3 \end{pmatrix}. \tag{284}$$

The tensor products between triplets are obtained as

$$\begin{pmatrix} x_1 \\ x_2 \\ x_4 \end{pmatrix}_{\mathbf{3}} \otimes \begin{pmatrix} y_1 \\ y_2 \\ y_4 \end{pmatrix}_{\mathbf{3}} = \begin{pmatrix} x_2y_4 \\ x_4y_1 \\ x_1y_2 \end{pmatrix}_{\bar{\mathbf{3}}} \oplus \begin{pmatrix} x_4y_2 \\ x_1y_4 \\ x_2y_1 \end{pmatrix}_{\bar{\mathbf{3}}} \oplus \begin{pmatrix} x_4y_4 \\ x_1y_1 \\ x_2y_2 \end{pmatrix}_{\mathbf{3}}, \tag{285}$$

$$\begin{pmatrix} x_6 \\ x_5 \\ x_3 \end{pmatrix}_{\bar{\mathbf{3}}} \otimes \begin{pmatrix} y_6 \\ y_5 \\ y_3 \end{pmatrix}_{\bar{\mathbf{3}}} = \begin{pmatrix} x_5y_3 \\ x_3y_6 \\ x_6y_5 \end{pmatrix}_{\mathbf{3}} \oplus \begin{pmatrix} x_3y_5 \\ x_6y_3 \\ x_5y_6 \end{pmatrix}_{\mathbf{3}} \oplus \begin{pmatrix} x_3y_3 \\ x_6y_6 \\ x_5y_5 \end{pmatrix}_{\bar{\mathbf{3}}}, \tag{286}$$

$$\begin{pmatrix} x_1 \\ x_2 \\ x_4 \end{pmatrix}_{\mathbf{3}} \otimes \begin{pmatrix} y_6 \\ y_5 \\ y_3 \end{pmatrix}_{\bar{\mathbf{3}}} = \begin{pmatrix} x_2y_6 \\ x_4y_5 \\ x_1y_3 \end{pmatrix}_{\mathbf{3}} \oplus \begin{pmatrix} x_1y_5 \\ x_2y_3 \\ x_4y_6 \end{pmatrix}_{\bar{\mathbf{3}}} \oplus \sum_{k=0,1,2} (x_1y_6 + \omega^k x_2y_5 + \omega^{2k} x_4y_3)_{\mathbf{1}_k}. \tag{287}$$

The tensor products between singlets are obtained as

$$\begin{aligned} &(x)_{\mathbf{1}_0}(y)_{\mathbf{1}_0} = (x)_{\mathbf{1}_1}(y)_{\mathbf{1}_2} = (x)_{\mathbf{1}_2}(y)_{\mathbf{1}_1} = (xy)_{\mathbf{1}_0}, \\ &(x)_{\mathbf{1}_1}(y)_{\mathbf{1}_1} = (xy)_{\mathbf{1}_2},\ (x)_{\mathbf{1}_2}(y)_{\mathbf{1}_2} = (xy)_{\mathbf{1}_1}. \end{aligned} \tag{288}$$

The tensor products between triplets and singlets are obtained as

$$(y)_{\mathbf{1}_k} \otimes \begin{pmatrix} x_{1(6)} \\ x_{2(5)} \\ x_{4(3)} \end{pmatrix}_{\mathbf{3}(\bar{\mathbf{3}})} = \begin{pmatrix} yx_{1(6)} \\ yx_{2(5)} \\ yx_{4(3)} \end{pmatrix}_{\mathbf{3}(\bar{\mathbf{3}})}. \tag{289}$$

11 $\Sigma(3N^3)$

11.1 Generic aspects

The discrete group $\Sigma(3N^3)$ is defined as a closed algebra of three Abelian symmetries, Z_N, Z'_N and Z''_N, which commute each other, and their Z_3 permutations. That is, when we denote the generators of Z_N, Z'_N and Z''_N by a, a' and a'', respectively, and the Z_3 generator is written by b, all of $\Sigma(3N^3)$ elements are written as

$$g = b^k a^m a'^n a''^\ell, \tag{290}$$

with $k = 0, 1, 2$, and $m, n, \ell = 0, ..., N-1$, where a, a', a'' and b satisfy the following relations:

$$\begin{aligned} &a^N = a'^N = a''^N = 1, \quad aa' = a'a, \quad aa'' = a''a, \quad a''a' = a'a'', \quad b^3 = 1, \\ &b^2ab = a'', \quad b^2a'b = a, \quad b^2a''b = a'. \end{aligned} \tag{291}$$

These generators, a, a', a'' and b, are represented, e.g, as

$$b = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, \quad a = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \rho \end{pmatrix}, \quad a' = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \rho & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad a'' = \begin{pmatrix} \rho & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \tag{292}$$

where $\rho = e^{2i\pi/N}$. Then, all of $\Sigma(3N^3)$ elements are written as

$$\begin{pmatrix} 0 & \rho^n & 0 \\ 0 & 0 & \rho^m \\ \rho^\ell & 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} \rho^\ell & 0 & 0 \\ 0 & \rho^m & 0 \\ 0 & 0 & \rho^n \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 & \rho^m \\ \rho^\ell & 0 & 0 \\ 0 & \rho^n & 0 \end{pmatrix}. \tag{293}$$

When $N = 2$, the element $aa'a''$ commutes with all of the elements. In addition, when we define $\tilde{a} = aa''$ and $\tilde{a}' = a'a''$, the closed algebra among $\tilde{a}$, $\tilde{a}'$ and b corresponds to $\Delta(12)$, where the element $aa'a''$ is not included. That is, this group is isomorphic to $Z_2 \times \Delta(12)$.

The situation for $N = 3$ is different. It is the same as the fact that the element $aa'a''$ commutes with all of the elements. Furthermore, when we define $\tilde{a} = a^2a''$ and $\tilde{a}' = a'a''^2$, the closed algebra among $\tilde{a}$, $\tilde{a}'$ and b corresponds to $\Delta(27)$. However, since the element $aa'a''$ is written by $aa'a'' = \tilde{a}^2\tilde{a}'$ in this case, the element is inside of $\Delta(27)$. Thus, the group $\Sigma(81)$ is not $Z_3 \times \Delta(27)$, but isomorphic to $(Z_3 \times Z'_3 \times Z''_3) \rtimes Z_3$.

Similarly, for generic value of N, the element $aa'a''$ commutes with all of the elements. When we define $\tilde{a} = a^{N-1}a''$ and $\tilde{a}' = a'a''^{N-1}$, the closed algebra among $\tilde{a}$, $\tilde{a}'$ and b corresponds to $\Delta(3N^2)$. When $N/3 \neq$ integer, the element $aa'a''$ is not included in $\Delta(3N^2)$. Thus, we find that this group is isomorphic to $Z_N \times \Delta(3N^2)$. On the other hand, when $N/3 =$ integer, the element $aa'a''$ is included in $\Delta(3N^2)$. That is, the group $\Sigma(3N^3)$ can not be $Z_N \times \Delta(3N^2)$. The group $\Sigma(3N^3)$ with $N/3 =$ integer has $N(N^2+8)/3$ conjugacy classes, $3N$ singlets, and $N(N^2-1)/3$ triplets.

11.2 $\Sigma(81)$

$\Sigma(81)$ has eighty-one elements and those are written as $b^k a^m a'^n a''^\ell$for $k = 0, 1, 2$ and $m, n, \ell = 0,\ 1,\ 2$, where $a,\ a',\ a''$, and b satisfy $a^3 = a'^3 = a''^3 = 1,\ aa' = a'a,\ aa'' = a''a,\ a''a' = a'a'',\ b^3 = 1,\ b^2ab = a'',\ b^2a'b = a$ and $b^2a''b = a'$. These elements are classified into seventeen conjugacy classes,

$$
\begin{array}{ccc}
C_1 : & \{e\}, & h = 1, \\
C_1^{(1)} : & \{aa'a''\}, & h = 3, \\
C_1^{(2)} : & \{(aa'a'')^2\}, & h = 3, \\
C_3^{0)} : & \{a^0a'^1a''^2\ ,\ a''^0a^1a'^2\ ,\ a'^0a''^1a^2\}, & h = 3, \\
C_3^{'(0)} : & \{a^1a'^0a''^2\ ,\ a''^1a^0a'^2\ ,\ a'^1a''^0a^2\}, & h = 3, \\
C_3^{(1)} : & \{a^0a'^0a''^1\ ,\ a''^0a^0a'^1\ ,\ a'^0a''^0a^1\}, & h = 3, \\
C_3^{'(1)} : & \{a^0a'^2a''^2\ ,\ a''^0a^2a'^2\ ,\ a'^0a''^2a^2\}, & h = 3, \\
C_3^{''(1)} : & \{a^1a'^1a''^2\ ,\ a''^1a^1a'^2\ ,\ a'^1a''^1a^2\}, & h = 3, \\
C_3^{(2)} : & \{a^0a'^0a''^2\ ,\ a''^0a^0a'^2\ ,\ a'^0a''^0a^2\}, & h = 3, \\
C_3^{'(2)} : & \{a^0a'^1a''^1\ ,\ a''^0a^1a'^1\ ,\ a'^0a''^1a^1\}, & h = 3, \\
C_3^{''(2)} : & \{a^1a'^2a''^2\ ,\ a''^1a^2a'^2\ ,\ a'^1a''^2a^2\}, & h = 3, \\
C_9^{0} : & \{ba^aa'^ba''^c\} & h = 3, \\
C_9^{1} : & \{ba^aa'^ba''^c\} & h = 9, \\
C_9^{2} : & \{ba^aa'^ba''^c\} & h = 9, \\
C_9^{'(0)} : & \{b^2a^aa'^ba''^c\}, & h = 3, \\
C_9^{'(1)} : & \{b^2a^aa'^ba''^c\}, & h = 9, \\
C_9^{''(2)} : & \{b^2a^aa'^ba''^c\}, & h = 9,
\end{array}
\tag{294}
$$

where we have shown also the orders of each element in the conjugacy class by h.

The relations (12) and (13) for $\Sigma(81)$ lead to

$$m_1 + 2^2 m_2 + 3^2 m_3 + \cdots = 81, \tag{295}$$

$$m_1 + m_2 + m_3 + \cdots = 17. \tag{296}$$

The solution is found as $(m_1, m_3) = (9, 8)$. That is, there are nine singlets $\mathbf{1}_\ell^k$ with $k, \ell = 0, 1, 2$ and eight triplets, $\mathbf{3}_A$, $\mathbf{3}_B$, $\mathbf{3}_C$, $\mathbf{3}_D$, $\bar{\mathbf{3}}_A$, $\bar{\mathbf{3}}_B$, $\bar{\mathbf{3}}_C$ and $\bar{\mathbf{3}}_D$. The character tables are given by Tables 24 and 25.

On all of the triplets, the generator b is represented as

$$b = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}. \tag{297}$$

The generators, a, a' and a'', are represented on each triplet as

$$a = \begin{pmatrix} \omega & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad a' = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \omega & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad a'' = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \omega \end{pmatrix}, \tag{298}$$

	h	$\chi_{1_0^0}$	$\chi_{1_1^0}$	$\chi_{1_2^0}$	$\chi_{1_0^1}$	$\chi_{1_1^1}$	$\chi_{1_2^1}$	$\chi_{1_0^2}$	$\chi_{1_1^2}$	$\chi_{1_2^2}$
C_1	1	1	1	1	1	1	1	1	1	1
$C_1^{(1)}$	1	1	1	1	1	1	1	1	1	1
$C_1^{(2)}$	1	1	1	1	1	1	1	1	1	1
$C_3^{(0)}$	3	1	1	1	1	1	1	1	1	1
$C_3^{\prime(0)}$	3	1	1	1	1	1	1	1	1	1
$C_3^{(1)}$	3	1	1	1	ω	ω	ω	ω^2	ω^2	ω^2
$C_3^{\prime(1)}$	3	1	1	1	ω	ω	ω	ω^2	ω^2	ω^2
$C_3^{\prime\prime(1)}$	3	1	1	1	ω	ω	ω	ω^2	ω^2	ω^2
$C_3^{(2)}$	3	1	1	1	ω^2	ω^2	ω^2	ω	ω	ω
$C_3^{\prime(2)}$	3	1	1	1	ω^2	ω^2	ω^2	ω	ω	ω
$C_3^{\prime\prime(2)}$	3	1	1	1	ω^2	ω^2	ω^2	ω	ω	ω
$C_9^{(0)}$	3	1	ω	ω^2	1	ω	ω^2	1	ω	ω^2
$C_9^{(1)}$	9	1	ω	ω^2	ω	ω^2	1	ω^2	1	ω
$C_9^{(2)}$	9	1	ω	ω^2	ω^2	1	ω	ω	ω^2	1
$C_9^{\prime(0)}$	3	1	ω^2	ω	1	ω^2	ω	1	ω^2	ω
$C_9^{\prime(1)}$	9	1	ω^2	ω	ω^2	ω	1	ω	1	ω^2
$C_9^{\prime(2)}$	9	1	ω^2	ω	ω	1	ω^2	ω^2	ω	1

Table 24: Characters of $\Sigma(81)$ for the 9 one-dimensional representations.

on $\mathbf{3}_A$,

$$a = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \omega^2 & 0 \\ 0 & 0 & \omega^2 \end{pmatrix}, \quad a' = \begin{pmatrix} \omega^2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \omega^2 \end{pmatrix}, \quad a'' = \begin{pmatrix} \omega^2 & 0 & 0 \\ 0 & \omega^2 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \tag{299}$$

on $\mathbf{3}_B$,

$$a = \begin{pmatrix} \omega^2 & 0 & 0 \\ 0 & \omega & 0 \\ 0 & 0 & \omega \end{pmatrix}, \quad a' = \begin{pmatrix} \omega & 0 & 0 \\ 0 & \omega^2 & 0 \\ 0 & 0 & \omega \end{pmatrix}, \quad a'' = \begin{pmatrix} \omega & 0 & 0 \\ 0 & \omega & 0 \\ 0 & 0 & \omega^2 \end{pmatrix}, \tag{300}$$

on $\mathbf{3}_C$,

$$a = \begin{pmatrix} \omega^2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \omega \end{pmatrix}, \quad a' = \begin{pmatrix} \omega & 0 & 0 \\ 0 & \omega^2 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad a'' = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \omega & 0 \\ 0 & 0 & \omega^2 \end{pmatrix}, \tag{301}$$

on $\mathbf{3}_D$. The representations of a, a' and a'' on $\mathbf{3}_{\bar{A}}$, $\mathbf{3}_{\bar{B}}$, $\mathbf{3}_{\bar{C}}$ and $\mathbf{3}_{\bar{D}}$ are obtained as complex conjugates of the representations on $\mathbf{3}_A$, $\mathbf{3}_B$, $\mathbf{3}_C$ and $\mathbf{3}_D$, respectively.

Class	n	h	χ_{3_A}	$\chi_{\bar{3}_A}$	χ_{3_B}	$\chi_{\bar{3}_B}$	χ_{3_C}	$\chi_{\bar{3}_C}$	χ_{3_D}	$\chi_{\bar{3}_D}$
C_1	1	1	3	3	3	3	3	3	3	3
$C_1^{(1)}$	1	3	3ω	$3\omega^2$	3ω	$3\omega^2$	3ω	$3\omega^2$	3	3
$C_1^{(2)}$	1	3	$3\omega^2$	3ω	$3\omega^2$	3ω	$3\omega^2$	3ω	3	3
$C_3^{(0)}$	3	3	0	0	0	0	0	0	3ω	$3\omega^2$
$C_3^{'(0)}$	3	3	0	0	0	0	0	0	$3\omega^2$	3ω
$C_3^{(1)}$	3	3	$-i\sqrt{3}$	$i\sqrt{3}$	$-i\sqrt{3}\omega^2$	$i\sqrt{3}\omega$	$-i\sqrt{3}\omega$	$i\sqrt{3}\omega^2$	0	0
$C_3^{(2)}$	3	3	$i\sqrt{3}$	$-i\sqrt{3}$	$i\sqrt{3}\omega$	$-i\sqrt{3}\omega^2$	$i\sqrt{3}\omega^2$	$-i\sqrt{3}\omega$	0	0
$C_3^{'(1)}$	3	3	$-i\sqrt{3}\omega^2$	$i\sqrt{3}\omega$	$-i\sqrt{3}\omega$	$i\sqrt{3}\omega^2$	$-i\sqrt{3}$	$i\sqrt{3}$	0	0
$C_3^{'(2)}$	3	3	$i\sqrt{3}\omega$	$-i\sqrt{3}\omega^2$	$i\sqrt{3}\omega^2$	$-i\sqrt{3}\omega$	$i\sqrt{3}$	$-i\sqrt{3}$	0	0
$C_3^{''(1)}$	3	3	$-i\sqrt{3}\omega$	$i\sqrt{3}\omega^2$	$-i\sqrt{3}$	$i\sqrt{3}$	$-i\sqrt{3}\omega^2$	$i\sqrt{3}\omega$	0	0
$C_3^{''(2)}$	3	3	$i\sqrt{3}\omega^2$	$-i\sqrt{3}\omega$	$i\sqrt{3}$	$-i\sqrt{3}$	$i\sqrt{3}\omega$	$-i\sqrt{3}\omega^2$	0	0
$C_9^{(0)}$	9	3	0	0	0	0	0	0	0	0
$C_9^{(1)}$	9	9	0	0	0	0	0	0	0	0
$C_9^{(2)}$	9	9	0	0	0	0	0	0	0	0
$C_9^{\prime(0)}$	9	3	0	0	0	0	0	0	0	0
$C_9^{\prime(1)}$	9	9	0	0	0	0	0	0	0	0
$C_9^{\prime(2)}$	9	9	0	0	0	0	0	0	0	0

Table 25: Characters of $\Sigma(81)$ for the 8 three-dimensional representations.

On the other hand, these generators are represented on the singlet $\mathbf{1}_\ell^k$ as $b = \omega^\ell$ and $a = a' = a'' = \omega^k$.

The tensor products between triplets are obtained as

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}_{\mathbf{3}_A} \otimes \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}_{\mathbf{3}_A} = \begin{pmatrix} x_1y_1 \\ x_2y_2 \\ x_3y_3 \end{pmatrix}_{\bar{\mathbf{3}}_A} \oplus \begin{pmatrix} x_2y_3 \\ x_3y_1 \\ x_1y_2 \end{pmatrix}_{\bar{\mathbf{3}}_B} \oplus \begin{pmatrix} x_3y_2 \\ x_1y_3 \\ x_2y_1 \end{pmatrix}_{\bar{\mathbf{3}}_B}, \tag{302}$$

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}_{\mathbf{3}_A} \otimes \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}_{\bar{\mathbf{3}}_A} = \left(\sum_{\ell=0,1,2} (x_1y_1 + \omega^{2\ell}x_2y_2 + \omega^{\ell}x_3y_3)_{\mathbf{1}_\ell^0}\right) \oplus \begin{pmatrix} x_3y_1 \\ x_1y_2 \\ x_2y_3 \end{pmatrix}_{\mathbf{3}_D} \oplus \begin{pmatrix} x_1y_3 \\ x_2y_1 \\ x_3y_2 \end{pmatrix}_{\bar{\mathbf{3}}_D}, \tag{303}$$

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}_{\mathbf{3}_A} \otimes \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}_{\mathbf{3}_B} = \begin{pmatrix} x_1y_1 \\ x_2y_2 \\ x_3y_3 \end{pmatrix}_{\bar{\mathbf{3}}_C} \oplus \begin{pmatrix} x_3y_2 \\ x_1y_3 \\ x_2y_1 \end{pmatrix}_{\bar{\mathbf{3}}_A} \oplus \begin{pmatrix} x_2y_3 \\ x_3y_1 \\ x_1y_2 \end{pmatrix}_{\bar{\mathbf{3}}_A}, \tag{304}$$

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}_{\mathbf{3}_A} \otimes \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}_{\bar{\mathbf{3}}_B} = \left(\sum_{\ell=0,1,2} (x_1y_1 + \omega^{2\ell}x_2y_2 + \omega^{\ell}x_3y_3)_{\mathbf{1}_\ell^2}\right) \oplus \begin{pmatrix} x_2y_3 \\ x_3y_1 \\ x_1y_2 \end{pmatrix}_{\mathbf{3}_D} \oplus \begin{pmatrix} x_2y_1 \\ x_3y_2 \\ x_1y_3 \end{pmatrix}_{\bar{\mathbf{3}}_D}, \tag{305}$$

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}_{\mathbf{3}_A} \otimes \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}_{\mathbf{3}_C} = \begin{pmatrix} x_1y_1 \\ x_2y_2 \\ x_3y_3 \end{pmatrix}_{\bar{\mathbf{3}}_B} \oplus \begin{pmatrix} x_2y_3 \\ x_3y_1 \\ x_1y_2 \end{pmatrix}_{\bar{\mathbf{3}}_C} \oplus \begin{pmatrix} x_3y_2 \\ x_1y_3 \\ x_2y_1 \end{pmatrix}_{\bar{\mathbf{3}}_C}, \tag{306}$$

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}_{\mathbf{3}_A} \otimes \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}_{\bar{\mathbf{3}}_C} = \left(\sum_{\ell=0,1,2} (x_1y_1 + \omega^{2\ell}x_2y_2 + \omega^{\ell}x_3y_3)_{\mathbf{1}_\ell^2}\right) \oplus \begin{pmatrix} x_2y_3 \\ x_3y_1 \\ x_1y_2 \end{pmatrix}_{\mathbf{3}_D} \oplus \begin{pmatrix} x_2y_1 \\ x_1y_2 \\ x_3y_3 \end{pmatrix}_{\bar{\mathbf{3}}_D}, \tag{307}$$

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}_{\bar{\mathbf{3}}_A} \otimes \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}_{\mathbf{3}_C} = \left(\sum_{\ell=0,1,2} (x_1y_1 + \omega^{2\ell}x_2y_2 + \omega^{\ell}x_3y_3)_{\mathbf{1}^1_\ell} \right) \oplus \begin{pmatrix} x_1y_2 \\ x_2y_3 \\ x_3y_1 \end{pmatrix}_{\mathbf{3}_D} \oplus \begin{pmatrix} x_3y_2 \\ x_1y_3 \\ x_2y_1 \end{pmatrix}_{\bar{\mathbf{3}}_D}, \tag{308}$$

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}_{\mathbf{3}_D} \otimes \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}_{\mathbf{3}_D} = \begin{pmatrix} x_1y_1 \\ x_2y_2 \\ x_3y_3 \end{pmatrix}_{\bar{\mathbf{3}}_D} \oplus \begin{pmatrix} x_2y_3 \\ x_3y_1 \\ x_1y_2 \end{pmatrix}_{\bar{\mathbf{3}}_D} \oplus \begin{pmatrix} x_3y_2 \\ x_1y_3 \\ x_2y_1 \end{pmatrix}_{\bar{\mathbf{3}}_D}, \tag{309}$$

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}_{\mathbf{3}_D} \otimes \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}_{\bar{\mathbf{3}}_D} = \sum_{\ell=0,1,2} [(x_1y_1 + \omega^{2\ell}x_2y_2 + \omega^{\ell}x_3y_3)_{\mathbf{1}^0_\ell} \oplus(x_2y_3 + \omega^{2\ell}x_3y_1 + \omega^{\ell}x_1y_2)_{\mathbf{1}^1_\ell} \oplus(x_3y_2 + \omega^{2\ell}x_1y_3 + \omega^{\ell}x_2y_1)_{\mathbf{1}^2_\ell}]. \tag{310}$$

The tensor products between singlets are obtained as

$$\mathbf{1}^k_\ell \otimes \mathbf{1}^{k'}_{\ell'} = \mathbf{1}^{k+k' \pmod 3}_{\ell+\ell' \pmod 3}. \tag{311}$$

The tensor products between singlets and triplets are obtained as

$$(x)_{\mathbf{1}^0_0} \otimes \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}_{\mathbf{3}(\bar{\mathbf{3}})_A} = \begin{pmatrix} xy_1 \\ xy_2 \\ xy_3 \end{pmatrix}_{\mathbf{3}(\bar{\mathbf{3}})_A}, \quad (x)_{\mathbf{1}^0_1} \otimes \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}_{\mathbf{3}(\bar{\mathbf{3}})_A} = \begin{pmatrix} xy_1 \\ \omega xy_2 \\ \omega^2 xy_3 \end{pmatrix}_{\mathbf{3}(\bar{\mathbf{3}})_A},$$
$$(x)_{\mathbf{1}^0_2} \otimes \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}_{\mathbf{3}(\bar{\mathbf{3}})_A} = \begin{pmatrix} xy_1 \\ \omega^2 xy_2 \\ \omega xy_3 \end{pmatrix}_{\mathbf{3}(\bar{\mathbf{3}})_A}, \tag{312}$$

$$(x)_{\mathbf{1}^1_0} \otimes \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}_{\mathbf{3}(\bar{\mathbf{3}})_A} = \begin{pmatrix} xy_1 \\ xy_2 \\ xy_3 \end{pmatrix}_{\mathbf{3}_C,(\bar{\mathbf{3}}_B)}, \quad (x)_{\mathbf{1}^1_1} \otimes \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}_{\mathbf{3}(\bar{\mathbf{3}})_A} = \begin{pmatrix} xy_1 \\ \omega xy_2 \\ \omega^2 xy_3 \end{pmatrix}_{\mathbf{3}_C,(\bar{\mathbf{3}}_B)},$$
$$(x)_{\mathbf{1}^1_2} \otimes \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}_{\mathbf{3}(\bar{\mathbf{3}})_A} = \begin{pmatrix} xy_1 \\ \omega^2 xy_2 \\ \omega xy_3 \end{pmatrix}_{\mathbf{3}_C,(\bar{\mathbf{3}}_B)}, \tag{313}$$

$$(x)_{\mathbf{1}^2_0} \otimes \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}_{\mathbf{3}(\bar{\mathbf{3}})_A} = \begin{pmatrix} xy_1 \\ xy_2 \\ xy_3 \end{pmatrix}_{\mathbf{3}_B,(\bar{\mathbf{3}}_C)}, \quad (x)_{\mathbf{1}^2_1} \otimes \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}_{\mathbf{3}(\bar{\mathbf{3}})_A} = \begin{pmatrix} xy_1 \\ \omega xy_2 \\ \omega^2 xy_3 \end{pmatrix}_{\mathbf{3}_B,(\bar{\mathbf{3}}_C)},$$
$$(x)_{\mathbf{1}^2_2} \otimes \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}_{\mathbf{3}(\bar{\mathbf{3}})_A} = \begin{pmatrix} xy_1 \\ \omega^2 xy_2 \\ \omega xy_3 \end{pmatrix}_{\mathbf{3}_B,(\bar{\mathbf{3}}_C)}, \tag{314}$$

$$(x)_{\mathbf{1}_0^0} \otimes \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}_{\mathbf{3}(\bar{\mathbf{3}})_B} = \begin{pmatrix} xy_1 \\ xy_2 \\ xy_3 \end{pmatrix}_{\mathbf{3}(\bar{\mathbf{3}})_B}, \ (x)_{\mathbf{1}_1^0} \otimes \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}_{\mathbf{3}(\bar{\mathbf{3}})_B} = \begin{pmatrix} xy_1 \\ \omega xy_2 \\ \omega^2 xy_3 \end{pmatrix}_{\mathbf{3}(\bar{\mathbf{3}})_B},$$

$$(x)_{\mathbf{1}_2^0} \otimes \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}_{\mathbf{3}(\bar{\mathbf{3}})_B} = \begin{pmatrix} xy_1 \\ \omega^2 xy_2 \\ \omega xy_3 \end{pmatrix}_{\mathbf{3}(\bar{\mathbf{3}})_B}, \tag{315}$$

$$(x)_{\mathbf{1}_0^1} \otimes \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}_{\mathbf{3}(\bar{\mathbf{3}})_B} = \begin{pmatrix} xy_1 \\ xy_2 \\ xy_3 \end{pmatrix}_{\mathbf{3}_A,(\bar{\mathbf{3}}_C)}, \ (x)_{\mathbf{1}_1^1} \otimes \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}_{\mathbf{3}(\bar{\mathbf{3}})_B} = \begin{pmatrix} xy_1 \\ \omega xy_2 \\ \omega^2 xy_3 \end{pmatrix}_{\mathbf{3}_A,(\bar{\mathbf{3}}_C)},$$

$$(x)_{\mathbf{1}_2^1} \otimes \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}_{\mathbf{3}(\bar{\mathbf{3}})_B} = \begin{pmatrix} xy_1 \\ \omega^2 xy_2 \\ \omega xy_3 \end{pmatrix}_{\mathbf{3}_A,(\bar{\mathbf{3}}_C)}, \tag{316}$$

$$(x)_{\mathbf{1}_0^2} \otimes \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}_{\mathbf{3}(\bar{\mathbf{3}})_B} = \begin{pmatrix} xy_1 \\ xy_2 \\ xy_3 \end{pmatrix}_{\mathbf{3}_C,(\bar{\mathbf{3}}_A)}, \ (x)_{\mathbf{1}_1^2} \otimes \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}_{\mathbf{3}(\bar{\mathbf{3}})_B} = \begin{pmatrix} xy_1 \\ \omega xy_2 \\ \omega^2 xy_3 \end{pmatrix}_{\mathbf{3}_C,(\bar{\mathbf{3}}_A)},$$

$$(x)_{\mathbf{1}_2^2} \otimes \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}_{\mathbf{3}(\bar{\mathbf{3}})_B} = \begin{pmatrix} xy_1 \\ \omega^2 xy_2 \\ \omega xy_3 \end{pmatrix}_{\mathbf{3}_C,(\bar{\mathbf{3}}_A)}, \tag{317}$$

$$(x)_{\mathbf{1}_0^0} \otimes \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}_{\mathbf{3}(\bar{\mathbf{3}})_C} = \begin{pmatrix} xy_1 \\ xy_2 \\ xy_3 \end{pmatrix}_{\mathbf{3}(\bar{\mathbf{3}})_C}, \ (x)_{\mathbf{1}_1^0} \otimes \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}_{\mathbf{3}(\bar{\mathbf{3}})_C} = \begin{pmatrix} xy_1 \\ \omega xy_2 \\ \omega^2 xy_3 \end{pmatrix}_{\mathbf{3}(\bar{\mathbf{3}})_C},$$

$$(x)_{\mathbf{1}_2^0} \otimes \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}_{\mathbf{3}(\bar{\mathbf{3}})_C} = \begin{pmatrix} xy_1 \\ \omega^2 xy_2 \\ \omega xy_3 \end{pmatrix}_{\mathbf{3}(\bar{\mathbf{3}})_C}, \tag{318}$$

$$(x)_{\mathbf{1}_0^1} \otimes \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}_{\mathbf{3}(\bar{\mathbf{3}})_C} = \begin{pmatrix} xy_1 \\ xy_2 \\ xy_3 \end{pmatrix}_{\mathbf{3}_B,(\bar{\mathbf{3}}_A)}, \ (x)_{\mathbf{1}_1^1} \otimes \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}_{\mathbf{3}(\bar{\mathbf{3}})_C} = \begin{pmatrix} xy_1 \\ \omega xy_2 \\ \omega^2 xy_3 \end{pmatrix}_{\mathbf{3}_B,(\bar{\mathbf{3}}_A)},$$

$$(x)_{\mathbf{1}_2^1} \otimes \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}_{\mathbf{3}(\bar{\mathbf{3}})_C} = \begin{pmatrix} xy_1 \\ \omega^2 xy_2 \\ \omega xy_3 \end{pmatrix}_{\mathbf{3}_B,(\bar{\mathbf{3}}_A)}, \tag{319}$$

$$(x)_{\mathbf{1}_0^2} \otimes \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}_{\mathbf{3}(\bar{\mathbf{3}})_C} = \begin{pmatrix} xy_1 \\ xy_2 \\ xy_3 \end{pmatrix}_{\mathbf{3}_A,(\bar{\mathbf{3}}_B)}, \ (x)_{\mathbf{1}_1^2} \otimes \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}_{\mathbf{3}(\bar{\mathbf{3}})_C} = \begin{pmatrix} xy_1 \\ \omega xy_2 \\ \omega^2 xy_3 \end{pmatrix}_{\mathbf{3}_A,(\bar{\mathbf{3}}_B)},$$

$$(x)_{\mathbf{1}_2^2} \otimes \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}_{\mathbf{3}(\bar{\mathbf{3}})_C} = \begin{pmatrix} xy_1 \\ \omega^2 xy_2 \\ \omega xy_3 \end{pmatrix}_{\mathbf{3}_A,(\bar{\mathbf{3}}_B)}, \tag{320}$$

$$(x)_{\mathbf{1}_\ell^k} \otimes \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}_{\mathbf{3}(\bar{\mathbf{3}})_D} = \begin{pmatrix} xy_1 \\ xy_2 \\ xy_3 \end{pmatrix}_{\mathbf{3}(\bar{\mathbf{3}})_D}, \tag{321}$$

where $k,\ \ell = 0,\ 1,\ 2.$

12 $\Delta(6N^2)$

12.1 Generic aspects

The discrete group $\Delta(6N^2)$ is isomorphic to $(Z_N \times Z'_N) \rtimes S_3$. (See also Ref. [196].) Its order is equal to $6N^2$. We denote the generators of Z_N and Z'_N by a and a', respectively, and the Z_3 and Z_2 generators of S_3 are written by b and c, respectively. They satisfy

$$\begin{aligned} & a^N = a'^N = b^3 = c^2 = (bc)^2 = e, \\ & aa' = a'a, \\ & bab^{-1} = a^{-1}a'^{-1}, \qquad ba'b^{-1} = a, \\ & cac^{-1} = a'^{-1}, \qquad ca'c^{-1} = a^{-1}. \end{aligned} \tag{322}$$

Using them, all of $\Delta(6N^2)$ elements are written as

$$b^k c^\ell a^m a'^n, \tag{323}$$

where $k = 0, 1, 2$, $\ell = 0, 1$ and $m, n = 0, \cdots, (N-1)$. Similarly to the previous sections, we can find conjugacy classes, characters, representations and tensor products of generic $\Delta(6N^2)$. Instead of showing those aspects for generic $\Delta(6N^2)$, we concentrate on an example, i.e. $\Delta(54)$.

12.2 $\Delta(54)$

The $\Delta(6)$ is nothing but S_3 and the $\Delta(24)$ is isomorphic to the S_4 group. Thus, the simple and non-trivial example is $\Delta(54)$.

- **Conjugacy classes**

All of the $\Delta(54)$ elements are written as $b^k c^\ell a^m a'^n$, where $k, m, n = 0, 1, 2$ and $\ell = 0, 1$. Half of them are the elements of $\Delta(27)$, whose conjugacy classes are shown in (275). Because of $cac^{-1} = a'^{-1}$ and $ca'c^{-1} = a^{-1}$, the conjugacy classes $C_1^{(1)}$ and $C_1^{(2)}$ of $\Delta(27)$ correspond to the conjugacy classes of $\Delta(54)$, still. However, the conjugacy classes $C_3^{(0,1)}$ and $C_3^{(0,1)}$ of $\Delta(27)$ are combined to a conjugacy class of $\Delta(54)$. Similarly, since $cba^k a'^\ell c^{-1} = b^2 ca^k a'^\ell c^{-1}$, the conjugacy classes $C_3^{(1,p)}$ and $C_3^{(2,p')}$ of $\Delta(27)$ for $p + p' = 0$ (mod 3) are combined to a conjugacy class of $\Delta(54)$.

Next, let us consider the conjugacy classes of elements including c. For example, we obtain

$$a^k a'^\ell (ca^m) a^{-k} a'^{-\ell} = ca^{m+p} a'^p, \tag{324}$$

where $p = -k - \ell$. Thus, the element ca^m is conjugate to $ca^{m+p}a'^p$ with $p = 0, 1, 2$. Furthermore, it is found that

$$b(ca^{m+p}a'^p)b^{-1} = b^2 ca^{-m} a'^{-m-p}, \tag{325}$$

$$b(ca^{-m}a'^{-m-p})b^{-1} = bca^{-p}a'^m. \tag{326}$$

Then, these elements belong to the same conjugacy class.

Using the above results, the $\Delta(54)$ elements are classified into the following conjugacy classes,

$$\begin{array}{rcl}
C_1: & \{e\}, & h=1, \\
C_1^{(1)}: & \{aa'^2\}, & h=3, \\
C_1^{(2)}: & \{a^2a'\}, & h=3, \\
C_6^{(0)}: & \{a', a, a^2a'^2, a'^2, a^2, aa'\}, & h=3, \\
C_6^{(1,0)}: & \{b, ba^2a', ba^1a'^2, b^2, b^2a^2a', b^2a^1a'^2\}, & h=3, \\
C_6^{(1,1)}: & \{ba, ba', ba^2a'^2, b^2a, b^2a', b^2a^2a'^2\}, & h=3, \\
C_6^{(1,2)}: & \{ba^2, ba^1a', ba'^2, b^2a^2, b^2a^1a', b^2a'^2\}, & h=3, \\
C_9^{(m)}: & \{ca^{m+p}a'^p, b^2ca^{-m}a'^{-m-p}, bca^{-p}a'^m \mid p=0,1,2\}, &
\end{array} \tag{327}$$

where $m = 0, 1, 2$. The total number of conjugacy classes is equal to ten. The relations (12) and (13) for $\Delta(54)$ lead to

$$m_1 + 2^2 m_2 + 3^2 m_3 + \cdots = 54, \tag{328}$$

$$m_1 + m_2 + m_3 + \cdots = 10. \tag{329}$$

The solution is found as $(m_1, m_2, m_3) = (2, 4, 4)$. That is, there are two singlets, four doublets and four triplets.

• Characters and representations

Now, let us study characters and representations. We start with two singlets. It is straightforward to find $\chi_{1\alpha}(a) = \chi_{1\alpha}(a') = \chi_{1\alpha}(b) = 1$ for two singlets from the above structure of conjugacy classes. In addition, because of $c^2 = e$, the two values ± 1 for $\chi_{1\pm}(c)$ are possible. They correspond to two singlets, $\mathbf{1}_\pm$.

Next, we study triplets. For example, the generators, a, a', b and c, are represented by

$$a = \begin{pmatrix} \omega^k & 0 & 0 \\ 0 & \omega^{2k} & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad a' = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \omega^k & 0 \\ 0 & 0 & \omega^{2k} \end{pmatrix},$$
$$b = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, \qquad c = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \tag{330}$$

on $\mathbf{3}_{1(k)}$ for $k = 1, 2$. Obviously, the $\Delta(54)$ algebra (322) is satisfied when we replace c by $-c$. That is, the generators, a, a', b and c, are represented by

$$a = \begin{pmatrix} \omega^k & 0 & 0 \\ 0 & \omega^{2k} & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad a' = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \omega^k & 0 \\ 0 & 0 & \omega^{2k} \end{pmatrix},$$
$$b = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, \qquad c = \begin{pmatrix} 0 & 0 & -1 \\ 0 & -1 & 0 \\ -1 & 0 & 0 \end{pmatrix}, \tag{331}$$

	χ_{1_+}	χ_{1_-}	$\chi_{3_{1(k)}}$	$\chi_{3_{2(k)}}$	χ_{2_1}	χ_{2_2}	χ_{2_3}	χ_{2_4}
C_1	1	1	3	3	2	2	2	2
$C_1^{(1)}$	1	1	$3\omega^k$	$3\omega^{2k}$	2	2	2	2
$C_1^{(2)}$	1	1	$3\omega^{2k}$	$3\omega^k$	2	2	2	2
$C_6^{(0)}$	1	1	0	0	2	-1	-1	-1
$C_6^{(1,0)}$	1	1	0	0	-1	-1	-1	2
$C_6^{(1,1)}$	1	1	0	0	-1	2	-1	-1
$C_6^{(1,2)}$	1	1	0	0	-1	-1	2	-1
$C_9^{(0)}$	1	-1	1	-1	0	0	0	0
$C_9^{(1)}$	1	-1	ω^{2k}	$-\omega^{2k}$	0	0	0	0
$C_9^{(2)}$	1	-1	ω^k	$-\omega^k$	0	0	0	0

Table 26: Characters of $\Delta(54)$

on $\mathbf{3}_{2(k)}$ for $k = 1, 2$. Then, characters χ_3 for $\mathbf{3}_{1(k)}$ and $\mathbf{3}_{2(k)}$ are shown in Table 26.

There are four doublets and the generators, a, a', b and c, are represented by

$$a = a' = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad b = \begin{pmatrix} \omega & 0 \\ 0 & \omega^2 \end{pmatrix}, \qquad c = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \text{on } \mathbf{2}_1, \tag{332}$$

$$a = a' = \begin{pmatrix} \omega^2 & 0 \\ 0 & \omega \end{pmatrix}, \qquad b = \begin{pmatrix} \omega & 0 \\ 0 & \omega^2 \end{pmatrix}, \qquad c = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \text{on } \mathbf{2}_2, \tag{333}$$

$$a = a' = \begin{pmatrix} \omega & 0 \\ 0 & \omega^2 \end{pmatrix}, \qquad b = \begin{pmatrix} \omega & 0 \\ 0 & \omega^2 \end{pmatrix}, \qquad c = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \text{on } \mathbf{2}_3, \tag{334}$$

$$a = a' = \begin{pmatrix} \omega & 0 \\ 0 & \omega^2 \end{pmatrix}, \qquad b = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad c = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \text{on } \mathbf{2}_4. \tag{335}$$

Then, characters χ_2 for $\mathbf{2}_{1,2,3,4}$ are shown in Table 26.

- **Tensor products**

The tensor products between triplets are obtained as

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}_{\mathbf{3}_{1(1)}} \otimes \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}_{\mathbf{3}_{1(1)}} = \begin{pmatrix} x_1y_1 \\ x_2y_2 \\ x_3y_3 \end{pmatrix}_{\mathbf{3}_{1(2)}} \oplus \begin{pmatrix} x_2y_3 + x_3y_2 \\ x_3y_1 + x_1y_3 \\ x_1y_2 + x_2y_1 \end{pmatrix}_{\mathbf{3}_{1(2)}} \oplus \begin{pmatrix} x_2y_3 - x_3y_2 \\ x_3y_1 - x_1y_3 \\ x_1y_2 - x_2y_1 \end{pmatrix}_{\mathbf{3}_{2(2)}}, \tag{336}$$

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}_{\mathbf{3}_{1(2)}} \otimes \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}_{\mathbf{3}_{1(2)}} = \begin{pmatrix} x_1y_1 \\ x_2y_2 \\ x_3y_3 \end{pmatrix}_{\mathbf{3}_{1(1)}} \oplus \begin{pmatrix} x_2y_3 + x_3y_2 \\ x_3y_1 + x_1y_3 \\ x_1y_2 + x_2y_1 \end{pmatrix}_{\mathbf{3}_{1(1)}} \oplus \begin{pmatrix} x_2y_3 - x_3y_2 \\ x_3y_1 - x_1y_3 \\ x_1y_2 - x_2y_1 \end{pmatrix}_{\mathbf{3}_{2(1)}}, \tag{337}$$

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}_{\mathbf{3}_{2(1)}} \otimes \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}_{\mathbf{3}_{2(1)}} = \begin{pmatrix} x_1y_1 \\ x_2y_2 \\ x_3y_3 \end{pmatrix}_{\mathbf{3}_{1(2)}} \oplus \begin{pmatrix} x_2y_3 + x_3y_2 \\ x_3y_1 + x_1y_3 \\ x_1y_2 + x_2y_1 \end{pmatrix}_{\mathbf{3}_{1(2)}} \oplus \begin{pmatrix} x_2y_3 - x_3y_2 \\ x_3y_1 - x_1y_3 \\ x_1y_2 - x_2y_1 \end{pmatrix}_{\mathbf{3}_{2(2)}} \tag{338}$$

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}_{\mathbf{3}_{2(2)}} \otimes \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}_{\mathbf{3}_{2(2)}} = \begin{pmatrix} x_1y_1 \\ x_2y_2 \\ x_3y_3 \end{pmatrix}_{\mathbf{3}_{1(1)}} \oplus \begin{pmatrix} x_2y_3 + x_3y_2 \\ x_3y_1 + x_1y_3 \\ x_1y_2 + x_2y_1 \end{pmatrix}_{\mathbf{3}_{1(1)}} \oplus \begin{pmatrix} x_2y_3 - x_3y_2 \\ x_3y_1 - x_1y_3 \\ x_1y_2 - x_2y_1 \end{pmatrix}_{\mathbf{3}_{2(1)}} \tag{339}$$

$$\begin{aligned}
\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}_{\mathbf{3}_{1(1)}} \otimes \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}_{\mathbf{3}_{1(2)}} = \; & (x_1y_1 + x_2y_2 + x_3y_3)_{\mathbf{1}_+} \oplus \begin{pmatrix} x_1y_1 + \omega^2 x_2y_2 + \omega x_3y_3 \\ \omega x_1y_1 + \omega^2 x_2y_2 + x_3y_3 \end{pmatrix}_{\mathbf{2}_1} \\
& \oplus \begin{pmatrix} x_1y_2 + \omega^2 x_2y_3 + \omega x_3y_1 \\ \omega x_1y_3 + \omega^2 x_2y_1 + x_3y_2 \end{pmatrix}_{\mathbf{2}_2} \oplus \begin{pmatrix} x_1y_3 + \omega^2 x_2y_1 + \omega x_3y_2 \\ \omega x_1y_2 + \omega^2 x_2y_3 + x_3y_1 \end{pmatrix}_{\mathbf{2}_3} \\
& \oplus \begin{pmatrix} x_1y_3 + x_2y_1 + x_3y_2 \\ x_1y_2 + x_2y_3 + x_3y_1 \end{pmatrix}_{\mathbf{2}_4},
\end{aligned} \tag{340}$$

$$\begin{aligned}
\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}_{\mathbf{3}_{1(1)}} \otimes \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}_{\mathbf{3}_{2(2)}} = \; & (x_1y_1 + x_2y_2 + x_3y_3)_{\mathbf{1}_-} \oplus \begin{pmatrix} x_1y_1 + \omega^2 x_2y_2 + \omega x_3y_3 \\ -\omega x_1y_1 - \omega^2 x_2y_2 - x_3y_3 \end{pmatrix}_{\mathbf{2}_1} \\
& \oplus \begin{pmatrix} x_1y_2 + \omega^2 x_2y_3 + \omega x_3y_1 \\ -\omega x_1y_3 - \omega^2 x_2y_1 - x_3y_2 \end{pmatrix}_{\mathbf{2}_2} \oplus \begin{pmatrix} x_1y_3 + \omega^2 x_2y_1 + \omega x_3y_2 \\ -\omega x_1y_2 - \omega^2 x_2y_3 - x_3y_1 \end{pmatrix}_{\mathbf{2}_3} \\
& \oplus \begin{pmatrix} x_1y_3 + x_2y_1 + x_3y_2 \\ -x_1y_2 - x_2y_3 - x_3y_1 \end{pmatrix}_{\mathbf{2}_4},
\end{aligned} \tag{341}$$

$$\begin{aligned}
\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}_{\mathbf{3}_{1(2)}} \otimes \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}_{\mathbf{3}_{2(1)}} = \; & (x_1y_1 + x_2y_2 + x_3y_3)_{\mathbf{1}_-} \oplus \begin{pmatrix} x_1y_1 + \omega^2 x_2y_2 + \omega x_3y_3 \\ -\omega x_1y_1 - \omega^2 x_2y_2 - x_3y_3 \end{pmatrix}_{\mathbf{2}_1} \\
& \oplus \begin{pmatrix} x_1y_3 + \omega^2 x_2y_1 + \omega x_3y_2 \\ -\omega x_1y_2 - \omega^2 x_2y_3 - x_3y_1 \end{pmatrix}_{\mathbf{2}_2} \oplus \begin{pmatrix} x_1y_2 + \omega^2 x_2y_3 + \omega x_3y_1 \\ -\omega x_1y_3 - \omega^2 x_2y_1 - x_3y_2 \end{pmatrix}_{\mathbf{2}_3} \\
& \oplus \begin{pmatrix} x_1y_2 + x_2y_3 + x_3y_1 \\ -x_1y_3 - x_2y_1 - x_3y_2 \end{pmatrix}_{\mathbf{2}_4}.
\end{aligned} \tag{342}$$

The tensor products between doublets are obtained as

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}_{\mathbf{2}_k} \otimes \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}_{\mathbf{2}_k} = (x_1y_2 + x_2y_1)_{\mathbf{1}_+} \oplus (x_1y_2 - x_2y_1)_{\mathbf{1}_-} \oplus \begin{pmatrix} x_2y_2 \\ x_1y_1 \end{pmatrix}_{\mathbf{2}_k}, \tag{343}$$

for $k = 1, 2, 3, 4$,

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}_{\mathbf{2}_1} \otimes \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}_{\mathbf{2}_2} = \begin{pmatrix} x_2y_2 \\ x_1y_1 \end{pmatrix}_{\mathbf{2}_3} \oplus \begin{pmatrix} x_2y_1 \\ x_1y_2 \end{pmatrix}_{\mathbf{2}_4}, \tag{344}$$

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}_{\mathbf{2}_1} \otimes \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}_{\mathbf{2}_3} = \begin{pmatrix} x_2 y_2 \\ x_1 y_1 \end{pmatrix}_{\mathbf{2}_2} \oplus \begin{pmatrix} x_2 y_1 \\ x_1 y_2 \end{pmatrix}_{\mathbf{2}_4}, \tag{345}$$

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}_{\mathbf{2}_1} \otimes \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}_{\mathbf{2}_4} = \begin{pmatrix} x_1 y_2 \\ x_2 y_1 \end{pmatrix}_{\mathbf{2}_2} \oplus \begin{pmatrix} x_1 y_1 \\ x_2 y_2 \end{pmatrix}_{\mathbf{2}_3}, \tag{346}$$

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}_{\mathbf{2}_2} \otimes \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}_{\mathbf{2}_3} = \begin{pmatrix} x_2 y_2 \\ x_1 y_1 \end{pmatrix}_{\mathbf{2}_1} \oplus \begin{pmatrix} x_1 y_2 \\ x_2 y_1 \end{pmatrix}_{\mathbf{2}_4}, \tag{347}$$

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}_{\mathbf{2}_2} \otimes \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}_{\mathbf{2}_4} = \begin{pmatrix} x_1 y_1 \\ x_2 y_2 \end{pmatrix}_{\mathbf{2}_1} \oplus \begin{pmatrix} x_1 y_2 \\ x_2 y_1 \end{pmatrix}_{\mathbf{2}_3}, \tag{348}$$

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}_{\mathbf{2}_3} \otimes \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}_{\mathbf{2}_4} = \begin{pmatrix} x_1 y_2 \\ x_2 y_1 \end{pmatrix}_{\mathbf{2}_1} \oplus \begin{pmatrix} x_1 y_1 \\ x_2 y_2 \end{pmatrix}_{\mathbf{2}_2}. \tag{349}$$

The tensor products between doublets and triplets are obtained as

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}_{\mathbf{2}_1} \otimes \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}_{\mathbf{3}_{1(k)}} = \begin{pmatrix} x_1 y_1 + \omega^2 x_2 y_1 \\ \omega x_1 y_2 + \omega x_2 y_2 \\ \omega^2 x_1 y_3 + x_2 y_3 \end{pmatrix}_{\mathbf{3}_{1(k)}} \oplus \begin{pmatrix} x_1 y_1 - \omega^2 x_2 y_1 \\ \omega x_1 y_2 - \omega x_2 y_2 \\ \omega^2 x_1 y_3 - x_2 y_3 \end{pmatrix}_{\mathbf{3}_{2(k)}}, \tag{350}$$

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}_{\mathbf{2}_1} \otimes \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}_{\mathbf{3}_{2(k)}} = \begin{pmatrix} x_1 y_1 + \omega^2 x_2 y_1 \\ \omega x_1 y_2 + \omega x_2 y_2 \\ \omega^2 x_1 y_3 + x_2 y_3 \end{pmatrix}_{\mathbf{3}_{2(k)}} \oplus \begin{pmatrix} x_1 y_1 - \omega^2 x_2 y_1 \\ \omega x_1 y_2 - \omega x_2 y_2 \\ \omega^2 x_1 y_3 - x_2 y_3 \end{pmatrix}_{\mathbf{3}_{1(k)}}, \tag{351}$$

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}_{\mathbf{2}_2} \otimes \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}_{\mathbf{3}_{1(1)}} = \begin{pmatrix} x_1 y_2 + \omega x_2 y_3 \\ \omega x_1 y_3 + \omega^2 x_2 y_1 \\ \omega^2 x_1 y_2 + x_2 y_2 \end{pmatrix}_{\mathbf{3}_{1(1)}} \oplus \begin{pmatrix} x_1 y_2 - \omega x_2 y_3 \\ \omega x_1 y_3 - \omega^2 x_2 y_1 \\ \omega^2 x_1 y_2 - x_2 y_2 \end{pmatrix}_{\mathbf{3}_{2(1)}}, \tag{352}$$

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}_{\mathbf{2}_2} \otimes \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}_{\mathbf{3}_{2(1)}} = \begin{pmatrix} x_1 y_2 + \omega x_2 y_3 \\ \omega x_1 y_3 + \omega^2 x_2 y_1 \\ \omega^2 x_1 y_2 + x_2 y_2 \end{pmatrix}_{\mathbf{3}_{2(1)}} \oplus \begin{pmatrix} x_1 y_2 - \omega x_2 y_3 \\ \omega x_1 y_3 - \omega^2 x_2 y_1 \\ \omega^2 x_1 y_2 - x_2 y_2 \end{pmatrix}_{\mathbf{3}_{1(1)}}, \tag{353}$$

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}_{\mathbf{2}_2} \otimes \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}_{\mathbf{3}_{1(2)}} = \begin{pmatrix} x_1 y_3 + \omega x_2 y_2 \\ \omega x_1 y_1 + \omega^2 x_2 y_3 \\ \omega^2 x_1 y_2 + x_2 y_1 \end{pmatrix}_{\mathbf{3}_{1(2)}} \oplus \begin{pmatrix} x_1 y_3 - \omega x_2 y_2 \\ \omega x_1 y_1 - \omega^2 x_2 y_2 \\ \omega^2 x_1 y_2 - x_2 y_1 \end{pmatrix}_{\mathbf{3}_{2(2)}}, \tag{354}$$

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}_{\mathbf{2}_2} \otimes \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}_{\mathbf{3}_{2(2)}} = \begin{pmatrix} x_1y_3 + \omega x_2y_2 \\ \omega x_1y_1 + \omega^2 x_2y_3 \\ \omega^2 x_1y_2 + x_2y_1 \end{pmatrix}_{\mathbf{3}_{2(2)}} \oplus \begin{pmatrix} x_1y_3 - \omega x_2y_2 \\ \omega x_1y_1 - \omega^2 x_2y_2 \\ \omega^2 x_1y_2 - x_2y_1 \end{pmatrix}_{\mathbf{3}_{1(2)}}, \tag{355}$$

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}_{\mathbf{2}_3} \otimes \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}_{\mathbf{3}_{1(1)}} = \begin{pmatrix} x_1y_3 + \omega x_2y_2 \\ \omega x_1y_1 + \omega^2 x_2y_3 \\ \omega^2 x_1y_2 + x_2y_1 \end{pmatrix}_{\mathbf{3}_{1(1)}} \oplus \begin{pmatrix} x_1y_3 - \omega x_2y_2 \\ \omega x_1y_1 - \omega^2 x_2y_2 \\ \omega^2 x_1y_2 - x_2y_1 \end{pmatrix}_{\mathbf{3}_{2(1)}}, \tag{356}$$

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}_{\mathbf{2}_3} \otimes \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}_{\mathbf{3}_{2(1)}} = \begin{pmatrix} x_1y_3 + \omega x_2y_2 \\ \omega x_1y_1 + \omega^2 x_2y_3 \\ \omega^2 x_1y_2 + x_2y_1 \end{pmatrix}_{\mathbf{3}_{2(1)}} \oplus \begin{pmatrix} x_1y_3 - \omega x_2y_2 \\ \omega x_1y_1 - \omega^2 x_2y_2 \\ \omega^2 x_1y_2 - x_2y_1 \end{pmatrix}_{\mathbf{3}_{1(1)}}, \tag{357}$$

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}_{\mathbf{2}_3} \otimes \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}_{\mathbf{3}_{1(2)}} = \begin{pmatrix} x_1y_2 + \omega x_2y_3 \\ \omega x_1y_3 + \omega^2 x_2y_1 \\ \omega^2 x_1y_1 + x_2y_2 \end{pmatrix}_{\mathbf{3}_{1(2)}} \oplus \begin{pmatrix} x_1y_2 - \omega x_2y_3 \\ \omega x_1y_3 - \omega^2 x_2y_1 \\ \omega^2 x_1y_1 - x_2y_2 \end{pmatrix}_{\mathbf{3}_{2(2)}}, \tag{358}$$

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}_{\mathbf{2}_3} \otimes \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}_{\mathbf{3}_{2(2)}} = \begin{pmatrix} x_1y_2 + \omega x_2y_3 \\ \omega x_1y_3 + \omega^2 x_2y_1 \\ \omega^2 x_1y_1 + x_2y_2 \end{pmatrix}_{\mathbf{3}_{2(2)}} \oplus \begin{pmatrix} x_1y_2 - \omega x_2y_3 \\ \omega x_1y_3 - \omega^2 x_2y_1 \\ \omega^2 x_1y_1 - x_2y_2 \end{pmatrix}_{\mathbf{3}_{1(2)}}, \tag{359}$$

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}_{\mathbf{2}_4} \otimes \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}_{\mathbf{3}_{1(1)}} = \begin{pmatrix} x_1y_3 + x_2y_2 \\ x_1y_1 + x_2y_3 \\ x_1y_2 + x_2y_1 \end{pmatrix}_{\mathbf{3}_{1(1)}} \oplus \begin{pmatrix} x_1y_3 - x_2y_2 \\ x_1y_1 - x_2y_3 \\ x_1y_2 - x_2y_1 \end{pmatrix}_{\mathbf{3}_{2(1)}}, \tag{360}$$

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}_{\mathbf{2}_4} \otimes \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}_{\mathbf{3}_{2(1)}} = \begin{pmatrix} x_1y_3 + x_2y_2 \\ x_1y_1 + x_2y_3 \\ x_1y_2 + x_2y_1 \end{pmatrix}_{\mathbf{3}_{2(1)}} \oplus \begin{pmatrix} x_1y_3 - x_2y_2 \\ x_1y_1 - x_2y_3 \\ x_1y_2 - x_2y_1 \end{pmatrix}_{\mathbf{3}_{1(1)}}, \tag{361}$$

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}_{\mathbf{2}_4} \otimes \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}_{\mathbf{3}_{1(2)}} = \begin{pmatrix} x_1y_2 + x_2y_3 \\ x_1y_3 + x_2y_1 \\ x_1y_1 + x_2y_2 \end{pmatrix}_{\mathbf{3}_{1(2)}} \oplus \begin{pmatrix} x_1y_2 - x_2y_3 \\ x_1y_3 - x_2y_1 \\ x_1y_1 - x_2y_2 \end{pmatrix}_{\mathbf{3}_{2(2)}}, \tag{362}$$

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}_{\mathbf{2}_4} \otimes \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}_{\mathbf{3}_{2(2)}} = \begin{pmatrix} x_1y_2 + x_2y_3 \\ x_1y_3 + x_2y_1 \\ x_1y_1 + x_2y_2 \end{pmatrix}_{\mathbf{3}_{2(2)}} \oplus \begin{pmatrix} x_1y_2 - x_2y_3 \\ x_1y_3 - x_2y_1 \\ x_1y_1 - x_2y_2 \end{pmatrix}_{\mathbf{3}_{1(2)}}. \tag{363}$$

Furthermore, the tensor products of the non-trivial singlet $\mathbf{1}_-$ with other representations are obtained as

$$\mathbf{2}_k \otimes \mathbf{1}_- = \mathbf{2}_k, \qquad \mathbf{3}_{1(k)} \otimes \mathbf{1}_- = \mathbf{3}_{2(k)}, \qquad \mathbf{3}_{2(k)} \otimes \mathbf{1}_- = \mathbf{3}_{1(k)}. \tag{364}$$

13 Subgroups and decompositions of multiplets

In the section 15, we see interesting applications of non-Abelian discrete symmetries for particle physics. For such applications, breaking of discrete symmetries is quite important, that is, breaking patterns of discrete groups and decompositions of multiplets. In this section, we study decompositions of multiplets for groups, which are studied in the previous sections. Suppose that a finite group G has the order N and M is a divisor of N. Then, Lagrange's theorem implies fine group H with the order M is a candidate for subgroups of G. (See Appendix A.)

An irreducible representation $\boldsymbol{r}_G$ of G can be decomposed in terms of irreducible representations $\boldsymbol{r}_{H,m}$ of its subgroup H as $\boldsymbol{r}_G = \sum_m \boldsymbol{r}_{H,m}$. If the trivial singlet of H is included in such a decomposition, $\sum_m \boldsymbol{r}_{H,m}$, and a scalar field with such a trivial singlet develops its vacuum expectation value (VEV), the group G breaks to H. On the other hand, if a scalar field in a multiplet $\boldsymbol{r}_G$ develops its VEV and it does not correspond to the trivial singlet of H, the group G breaks not to H, but to another group.

Furthermore, when we know group-theoretical aspects such as representations of G, it would be useful to use them to study those for subgroups of G.

In what follows, we show decompositions of multiplets of G into multiplets of subgroups. For a finite group G, there are several chains of subgroups, $G \to G_1 \to \cdots \to G_k \to Z_N \to \{e\}$, $G \to G'_1 \to \cdots \to G'_m \to Z_M \to \{e\}$, etc. It would be obvious that the smallest non-trivial subgroup in those chains is an Abelian group such as Z_N or Z_M. In most of cases, we concentrate on subgroups, which are shown explicitly in the previous sections. Then, we show the largest subgroup such as G_1 and G'_1 in each chain of subgroups.

13.1 S_3

Here, we start with S_3, because S_3 is the minimal non-Abelian discrete group. Its order is equal to $2 \times 3 = 6$. Thus, there are two candidates for subgroups. One is a group with the order two, and the other has the order three. The former corresponds to Z_2 and the latter corresponds to Z_3. As in section 3.1, the S_3 consists of $\{e, a, b, ab, ba, bab\}$, where $a^2 = e$ and $(ab)^3$=e. Indeed, the subgroup Z_2 consists of e.g. $\{e, a\}$, while the other combinations such as $\{e, b\}$ and $\{e, bab\}$ also correspond to Z_2. The subgroup Z_3 consists of $\{e, ab, ba = (ab)^2\}$. The S_3 has two singlets, $\mathbf{1}$ and $\mathbf{1}'$ and one doublet $\mathbf{2}$. Both of subgroups, Z_2 and Z_3, are Abelian. Thus, decompositions of multiplets under Z_2 and Z_3 are rather simple. We show such decompositions in what follows.

- $S_3 \to Z_3$

The following elements

$$\{e, ab, ba\},$$

of S_3 construct the Z_3 subgroup, which is the normal subgroup. There is no other choice to make a Z_3 subgroup. There are three singlet representations, $\mathbf{1}_k$ $k = 0, 1, 2$ for Z_3, that is, $ab = \omega^k$ on $\mathbf{1}_k$. Recall that $\chi_{\mathbf{1}}(ab) = \chi_{\mathbf{1}'}(ab) = 1$ for both $\mathbf{1}$ and $\mathbf{1}'$ of S_3. Thus, both $\mathbf{1}$

and $\mathbf{1}'$ of S_3 correspond to $\mathbf{1}_0$ of Z_3. On the other hand, the doublet $\mathbf{2}$ of S_3 decomposes into two singlets of Z_3. Since $\chi_2(ab) = -1$, the S_3 doublet $\mathbf{2}$ decomposes into $\mathbf{1}_1$ and $\mathbf{1}_2$ of Z_3.

In order to see this, we use the two dimensional representations of the group element ab (33),

$$ab = \begin{pmatrix} -\frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix}. \tag{365}$$

Then the doublet (x_1, x_2) decompose into two non-trivial singlets as

$$\mathbf{1}_1 : x_1 - ix_2 \qquad \mathbf{1}_2 : x_1 + ix_2. \tag{366}$$

- $S_3 \to Z_2$

We consider the Z_2 subgroup of S_3, which consists of e.g.

$$\{e, a\}.$$

There are two singlet representations $\mathbf{1}_k$ $k = 0, 1$, for Z_2, that is, $a = (-1)^k$ on $\mathbf{1}_k$. Recall that $\chi_1(a) = 1$ and $\chi_{1'}(a) = -1$ for $\mathbf{1}$ and $\mathbf{1}'$ of S_3. Thus, $\mathbf{1}$ and $\mathbf{1}'$ of S_3 correspond to $\mathbf{1}_0$ and $\mathbf{1}_1$ of Z_2, respectively. On the other hand, the doublet $\mathbf{2}$ of S_3 decomposes into two singlets of Z_2. Since $\chi_2(a) = -1$, the S_3 doublet $\mathbf{2}$ decomposes into $\mathbf{1}_0$ and $\mathbf{1}_1$ of Z_2. Indeed, the element a is represented on $\mathbf{2}$ in (33) as

$$a = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{367}$$

Then, for the doublet (x_1, x_2), the elements x_1 and x_2 correspond to $x_1 = \mathbf{1}_0$ and $x_2 = \mathbf{1}_1$, respectively.

In addition to $\{e, a\}$, there are other Z_2 subgroups, $\{e, b\}$ and $\{e, aba\}$. In both cases, the same results are obtained when we choose a proper basis. This is an example of Abelian subgroups. In non-Abelian subgroups, the same situation happens. That is, different elements of a finite group G construct the same subgroup. A simple example is D_6. All of the D_6 elements are written by $a^m b^k$ for $m = 0, 1, \cdots, 5$ and $k = 0, 1$, where $a^6 = e$ and $bab = a^{-1}$. Here, we denote $\tilde{a} = a^2$. Then, the elements $\tilde{a}^m b^k$ for $m = 0, 1, 2$ and $k = 0, 1$ correspond to the subgroup $D_3 \simeq S_3$. On the other hand, we denote $\tilde{b} = ab$. Then, the elements $\tilde{a}^m \tilde{b}^k$ for $m = 0, 1, 2$ and $k = 0, 1$ correspond to another D_3 subgroup. The decompositions of D_6 multiplets into D_3 multiplets are the same between both D_3 subgroups when we change a proper basis.

13.2 S_4

As mentioned in section 12, the S_4 group is isomorphic to $\Delta(24)$ and $(Z_2 \times Z_2) \rtimes S_3$. It would be convenient to use the terminology of $(Z_2 \times Z_2) \rtimes S_3$. That is, all of the

	$\mathbf{1}$	$\mathbf{1'}$	$\mathbf{2}$	$\mathbf{3}$	$\mathbf{3'}$
b	1	1	$\begin{pmatrix} \omega & 0 \\ 0 & \omega^2 \end{pmatrix}$	$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$
c	1	-1	$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & -1 \\ 0 & -1 & 0 \\ -1 & 0 & 0 \end{pmatrix}$
a	1	1	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$	$\begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$	$\begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$
a'	1	1	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$

Table 27: Representations of S_4 elements

elements are written as $b^k c^\ell a^m a'^{\,n}$ with $k = 0, 1, 2$ and $\ell, m, n = 0, 1$. (See section 12.) The generators, a, a', b and c, are related with the notation in section 3.2 as follows,

$$b = c_1, \qquad c = f_1, \qquad a = a_4, \qquad a' = a_2. \tag{368}$$

They satisfy the following algebraic relations

$$\begin{aligned} &b^3 = c^2 = (bc)^2 = a^2 = a'^2 = e, \quad aa' = a'a, \\ &bab^{-1} = a^{-1}a'^{-1}, \quad ba'b^{-1} = a, \quad cac^{-1} = a'^{-1}, \quad ca'a^{-1} = a^{-1}. \end{aligned} \tag{369}$$

Furthermore, their representations on $\mathbf{1}$, $\mathbf{1'}$, $\mathbf{2}$, $\mathbf{3}$ and $\mathbf{3'}$ are shown in Table 27. As subgroups, the S_4 includes non-Abelian groups, S_3, A_4 and $\Sigma(8)$, which is $(Z_2 \times Z_2) \rtimes Z_2$. Thus, the decompositions of S_4 are non-trivial compared with those of S_3.

• $S_4 \to S_3$

The subgroup S_3 elements are $\{a_1, b_1, d_1, d_1, e_1, f_1\}$. Alternatively, they are denoted by $b^k c^\ell$ with $k = 0, 1, 2$ and $\ell = 0, 1$, i.e., $\{e, b, b^2, c, bc, b^2c\}$. Among them, Table 28 shows the representations of the generators b and c on $\mathbf{1}$, $\mathbf{1'}$ and $\mathbf{2}$ of S_3. Then each representation of S_4 is decomposed as

$$\begin{array}{cccccc} S_4 & \mathbf{1} & \mathbf{1'} & \mathbf{2} & \mathbf{3} & \mathbf{3'} \\ & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\ S_3 & \mathbf{1} & \mathbf{1'} & \mathbf{2} & \mathbf{1}+\mathbf{2} & \mathbf{1'}+\mathbf{2} \end{array} . \tag{370}$$

The components of $\mathbf{3}$ (x_1, x_2, x_3) are decomposed to $\mathbf{1}$ and $\mathbf{2}$ as

$$\mathbf{1} : (x_1 + x_2 + x_3), \qquad \mathbf{2} : \begin{pmatrix} x_1 + \omega^2 x_2 + \omega x_3 \\ \omega(x_1 + \omega^2 x_2 + \omega x_3) \end{pmatrix}, \tag{371}$$

<table>
<tr><th></th><th>$\mathbf{1}$</th><th>$\mathbf{1}'$</th><th>$\mathbf{2}$</th></tr>
<tr><td>b</td><td>1</td><td>1</td><td>$\begin{pmatrix} \omega & 0 \\ 0 & \omega^2 \end{pmatrix}$</td></tr>
<tr><td>c</td><td>1</td><td>-1</td><td>$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$</td></tr>
</table>

Table 28: Representations of S_3 elements

<table>
<tr><th></th><th>$\mathbf{1}$</th><th>$\mathbf{1}'$</th><th>$\mathbf{1}''$</th><th>$\mathbf{3}$</th></tr>
<tr><td>b</td><td>1</td><td>ω</td><td>ω^2</td><td>$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$</td></tr>
<tr><td>a</td><td>1</td><td>1</td><td>1</td><td>$\begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$</td></tr>
<tr><td>a'</td><td>1</td><td>1</td><td>1</td><td>$\begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$</td></tr>
</table>

Table 29: Representations of A_4 elements

and components of $\mathbf{3}'$ are decomposed to $\mathbf{1}'$ and $\mathbf{2}$ as

$$\mathbf{1}' : (x_1 + x_2 + x_3), \qquad \mathbf{2} : \begin{pmatrix} x_1 + \omega^2 x_2 + \omega x_3 \\ -\omega(x_1 + \omega^2 x_2 + \omega x_3) \end{pmatrix} . \tag{372}$$

• $S_4 \to A_4$

The A_4 subgroup consists of $b^k a^m a'^n$ with $k = 0, 1, 2$ and $m, n = 0, 1$. Recall that the A_4 is isomorphic to $\Delta(12)$. Table 29 shows the representations of the generators b, a and a' on $\mathbf{1}$, $\mathbf{1}'$, $\mathbf{1}''$ and $\mathbf{3}$ of A_4. Then each representation of S_4 is decomposed as

$$\begin{array}{cccccc} S_4 & \mathbf{1} & \mathbf{1}' & \mathbf{2} & \mathbf{3} & \mathbf{3}' \\ & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\ A_4 & \mathbf{1} & \mathbf{1} & \mathbf{1}' + \mathbf{1}'' & \mathbf{3} & \mathbf{3} \end{array} . \tag{373}$$

• $S_4 \to \Sigma(8)$

The subgroup $\Sigma(8)$, i.e. $(Z_2 \times Z_2) \rtimes Z_2$, consists of $c^\ell a^m a'^n$ with $\ell, m, n = 0, 1$. Table 30 shows the representations of the generators c, a and a' on $\mathbf{1}_{+0}$, $\mathbf{1}_{+1}$, $\mathbf{1}_{-0}$, $\mathbf{1}_{-1}$ and $\mathbf{2}_{1,0}$ of $\Sigma(8)$. Then each representation of S_4 is decomposed as

$$\begin{array}{cccccc} S_4 & \mathbf{1} & \mathbf{1}' & \mathbf{2} & \mathbf{3} & \mathbf{3}' \\ & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\ \Sigma(8) & \mathbf{1}_{+0} & \mathbf{1}_{-0} & \mathbf{1}_{+0} + \mathbf{1}_{-0} & \mathbf{1}_{+1} + \mathbf{2} & \mathbf{1}_{-1} + \mathbf{2} \end{array} . \tag{374}$$

	$\mathbf{1}_{+0}$	$\mathbf{1}_{+1}$	$\mathbf{1}_{-0}$	$\mathbf{1}_{-1}$	$\mathbf{2}_{1,0}$
c	1	1	-1	-1	$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$
a	1	-1	1	-1	$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$
a'	1	-1	1	-1	$\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$

Table 30: Representations of $\Sigma(8)$ elements

The components of $\mathbf{3}$ (x_1, x_2, x_3) are decomposed to $\mathbf{1}_{+1}$ and $\mathbf{2}$ as

$$\mathbf{1}_{+1} : x_2, \qquad \mathbf{2} : \begin{pmatrix} x_3 \\ x_1 \end{pmatrix}, \tag{375}$$

and the components of $\mathbf{3}'$ are decomposed to $\mathbf{1}_{-1}$ and $\mathbf{2}$ as

$$\mathbf{1}_{-1} : x_2, \qquad \mathbf{2} : \begin{pmatrix} x_3 \\ -x_1 \end{pmatrix}. \tag{376}$$

13.3 A_5

All of the A_5 elements are written by products of $s = a$ and $t = bab$ as shown in section 4.2.

- $A_5 \to A_4$

The subgroup A_4 elements are $\{e, b, \tilde{a}, b\tilde{a}b^2, b^2\tilde{a}b, b\tilde{a}, \tilde{a}b, \tilde{a}b\tilde{a}, b^2\tilde{a}, b^2\tilde{a}b\tilde{a}b\}$ where $\tilde{a} = ab^2aba$. We denote $\tilde{t} = b$ and $\tilde{s} = \tilde{a}$. They satisfy the following relations

$$\tilde{s}^2 = \tilde{t}^3 = (\tilde{s}\tilde{t})^3 = e, \tag{377}$$

and correspond to the generators, s and t, of the A_4 group in section 4.1. Each representation of A_5 is decomposed as

$$\begin{array}{cccccc} A_5 & \mathbf{1} & \mathbf{3} & \mathbf{3}' & \mathbf{4} & \mathbf{5} \\ & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\ A_4 & \mathbf{1} & \mathbf{3} & \mathbf{3} & \mathbf{1}+\mathbf{3} & \mathbf{1}'+\mathbf{1}''+\mathbf{3} \end{array}. \tag{378}$$

- $A_5 \to D_5$

The D_5 subgroup consists of $a^k\tilde{a}^m$ with $k = 0, 1$ and $m = 0, 1, 2, 3, 4$ where $\tilde{a} \equiv bab^2a$. They satisfy $a^2 = \tilde{a}^5 = e$ and $a\tilde{a}a = \tilde{a}^4$. In order to identify the D_5 basis used in section 6, we define $\tilde{b} = abab^2a$. Table 31 shows the representations of these generators $\tilde{a}$, $\tilde{b}$ on

	$\mathbf{1}_+$	$\mathbf{1}_-$	$\mathbf{2}_1$	$\mathbf{2}_2$
$\tilde{a}$	1	1	$\begin{pmatrix} \exp 2\pi i/5 & 0 \\ 0 & -\exp 2\pi i/5 \end{pmatrix}$	$\begin{pmatrix} \exp 4\pi i/5 & 0 \\ 0 & -\exp 4\pi i/5 \end{pmatrix}$
$\tilde{b}$	1	-1	$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

Table 31: Representations of D_5 elements.

	1	**1**’	**1**”	**2**	**2**’	**2**”	**3**
s	1	1	1	$\frac{i}{\sqrt{3}}\begin{pmatrix} 1 & \sqrt{2} \\ \sqrt{2} & -1 \end{pmatrix}$	$\frac{i}{\sqrt{3}}\begin{pmatrix} 1 & \sqrt{2} \\ \sqrt{2} & -1 \end{pmatrix}$	$\frac{i}{\sqrt{3}}\begin{pmatrix} 1 & \sqrt{2} \\ \sqrt{2} & -1 \end{pmatrix}$	$\frac{1}{3}\begin{pmatrix} -1 & 2 & 2 \\ 2 & -1 & 2 \\ 2 & 2 & -1 \end{pmatrix}$
r	1	1	1	$\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$	$\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$	$\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$
t	1	ω	ω^2	$\begin{pmatrix} \omega & 0 \\ 0 & \omega^2 \end{pmatrix}$	$\begin{pmatrix} \omega^2 & 0 \\ 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 \\ 0 & \omega \end{pmatrix}$	$\frac{1}{3}\begin{pmatrix} 1 & 0 & 0 \\ 0 & \omega & 0 \\ 0 & 0 & \omega^2 \end{pmatrix}$

Table 32: Representations of T'

$\mathbf{1}_+$, $\mathbf{1}_-$, $\mathbf{2_1}$ and $\mathbf{2_2}$ of D_5. Then each representation of A_5 is decomposed as

$$\begin{array}{cccccc} A_5 & \mathbf{1} & \mathbf{3} & \mathbf{3}' & \mathbf{4} & \mathbf{5} \\ & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\ D_5 & \mathbf{1}_+ & \mathbf{1}_- + \mathbf{2_1} & \mathbf{1}_- + \mathbf{2_2} & \mathbf{2_1} + \mathbf{2_2} & \mathbf{1}_+ + \mathbf{2_1} + \mathbf{2_2} \end{array} . \tag{379}$$

• $A_5 \to S_3 \simeq D_3$

Recall that the S_3 group is isomorphic to the D_3 group. The subgroup D_3 consists of $b^k\tilde{a}^m$ with $k = 0, 1, 2$ and $m = 0, 1$ where we define $\tilde{a} = ab^2ab^2ab$. These generators satisfy $\tilde{a}^2 = e$ and $\tilde{a}b\tilde{a} = b^2$. Then each representation of A_5 is decomposed as

$$\begin{array}{cccccc} A_5 & \mathbf{1} & \mathbf{3} & \mathbf{3}' & \mathbf{4} & \mathbf{5} \\ & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\ D_3 & \mathbf{1}_+ & \mathbf{1}_- + \mathbf{2} & \mathbf{1}_- + \mathbf{2} & \mathbf{1}_+ + \mathbf{1}_- + \mathbf{2} & \mathbf{1}_+ + \mathbf{2} + \mathbf{2} \end{array} . \tag{380}$$

13.4 T'

All of the T' elements are written in terms of the generators, s and t as well as r, which satisfy the algebraic relations, $s^2 = r$, $r^2 = t^3 = (st)^3 = e$ and $rt = tr$. Table 32 shows the representations of s, t and r on each representation.

• $T' \to Z_6$

The subgroup Z_6 consists of a^m, with $m = 0, \cdots, 5$, where $a = rt$ and $a^6 = e$. The Z_6 group has six singlet representations, $\mathbf{1}_n$ with $n = 0, \cdots, 5$. On the singlet $\mathbf{1}_n$, the generator a is represented as $a = e^{2\pi i n/6}$. Thus, each representation of T' is decomposed as

$$\begin{array}{cccccccc} T' & \mathbf{1} & \mathbf{1}' & \mathbf{1}'' & \mathbf{2} & \mathbf{2}' & \mathbf{2}'' & \mathbf{3} \\ & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\ Z_6 & \mathbf{1}_0 & \mathbf{1}_2 & \mathbf{1}_4 & \mathbf{1}_5 + \mathbf{1}_1 & \mathbf{1}_5 + \mathbf{1}_3 & \mathbf{1}_3 + \mathbf{1}_5 & \mathbf{1}_0 + \mathbf{1}_2 + \mathbf{1}_4 \end{array} . \tag{381}$$

• $T' \to Z_4$

The subgroup Z_4 consists of $\{e, s, s^2, s^3\}$. The Z_4 group has two singlet representations, $\mathbf{1}_m$ with $m = 0, 1, 2, 3$. On the singlet $\mathbf{1}_m$, the generator s is represented as $s = e^{\pi i m/2}$. All of the doublets of T', $\mathbf{2}$, $\mathbf{2}'$, $\mathbf{2}''$, are decomposed to two singlets of Z_4 $\mathbf{1}_1$ and $\mathbf{1}_3$ as $\mathbf{1}_1 : \frac{1+\sqrt{3}}{\sqrt{2}} x_1 + x_2$ and $\mathbf{1}_3 : -\frac{-1+\sqrt{3}}{\sqrt{2}} i x_1 + x_2$, where (x_1, x_2) correspond to the doublets. In addition, the triplet $\mathbf{3} : (x_1, x_2, x_3)$ is decomposed to singlets, $\mathbf{1}_0 + \mathbf{1}_2 + \mathbf{1}_2$ as $\mathbf{1}_0 : (x_1 + x_2 + x_3)$, $\mathbf{1}_2 : (-x_1 + x_3)$ and $\mathbf{1}_2 : (-x_1 + x_2)$. The results are summarized as

$$\begin{array}{cccccccc} T' & \mathbf{1} & \mathbf{1}' & \mathbf{1}'' & \mathbf{2} & \mathbf{2}' & \mathbf{2}'' & \mathbf{3} \\ & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\ Z_4 & \mathbf{1}_0 & \mathbf{1}_0 & \mathbf{1}_0 & \mathbf{1}_1 + \mathbf{1}_3 & \mathbf{1}_1 + \mathbf{1}_3 & \mathbf{1}_1 + \mathbf{1}_3 & \mathbf{1}_0 + \mathbf{1}_2 + \mathbf{1}_2 \end{array} . \tag{382}$$

• $T' \to Q_4$

We consider the subgroup Q_4, which consists of $s^m b^k$ with $m = 0, 1, 2, 3$ and $k = 0, 1$. The generator b is defined by $b = tst^2$. Then, each representation of T' is decomposed as

$$\begin{array}{cccccccc} T' & \mathbf{1} & \mathbf{1}' & \mathbf{1}'' & \mathbf{2} & \mathbf{2}' & \mathbf{2}'' & \mathbf{3} \\ & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\ Q_4 & \mathbf{1}_{++} & \mathbf{1}_{++} & \mathbf{1}_{++} & \mathbf{2} & \mathbf{2} & \mathbf{2} & \mathbf{1}_{+-} + \mathbf{1}_{-+} + \mathbf{1}_{--} \end{array} . \tag{383}$$

13.5 D_4

Here, we study D_4, which is the second minimum discrete symmetry. All of the D_4 elements are written by $a^m b^k$ with $m = 0, 1, 2, 3$ and $k = 0, 1$. Since the order of D_4 is 8, it contains the order 2 and 4 subgroups. There are two types of the order 4 groups which are corresponding to $Z_2 \times Z_2$ and Z_4 groups. All of subgroups are Abelian. Thus, decompositions are rather simple.

• $D_4 \to Z_4$

The subgroup Z_4 is consist of the elements $\{e, a, a^2, a^3\}$. Obviously it is the normal subgroups of D_4 and there are four types of irreducible singlet representations $\mathbf{1}_m$ with $m = 0, 1, 2, 3$, where a is represented as $a = e^{\pi i m/2}$. From the characters of D_4 groups, it is found that $\mathbf{1}_{++}$ and $\mathbf{1}_{--}$ of D_4 correspond to $\mathbf{1}_0$ of Z_4 and $\mathbf{1}_{+-}$ and $\mathbf{1}_{-+}$ of D_4 correspond to $\mathbf{1}_2$ of D_4. For the D_4 doublet $\mathbf{2}$, it is convenient to use the diagonal base of matrix a as

$$a = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}. \tag{384}$$

Then we can read the doublet $\mathbf{2} : (x_1, x_2)$ decomposes to two singlets as $\mathbf{1}_1 : x_1$ and $\mathbf{1}_3 : x_2$.

• $D_4 \to Z_2 \times Z_2$

We denote $\tilde{a} = a^2$. Then, the subgroup $Z_2 \times Z_2$ consists of $\{e, \tilde{a}, b, \tilde{a}b\}$, where $\tilde{a}b = b\tilde{a}$ and $\tilde{a}^2 = b^2 = e$. Obviously, their representations are quite simple, that is, $\mathbf{1}_{\pm\pm}$, whose $Z_2 \times Z_2$ charges are determined by $\tilde{a} = \pm 1$ and $b = \pm 1$. We use the notation that the first (second) subscript of $\mathbf{1}_{\pm\pm}$ denotes the Z_2 charge for $\tilde{a}$ (b). Then, the singlets $\mathbf{1}_{++}$ and $\mathbf{1}_{+-}$ of D_4 correspond to $\mathbf{1}_{++}$ of $Z_2 \times Z_2$ and $\mathbf{1}_{-+}$ and $\mathbf{1}_{--}$ of D_4 correspond to $\mathbf{1}_{+-}$ of $Z_2 \times Z_2$. The doublet $\mathbf{2}$ of D_4 decomposes to $\mathbf{1}_{-+}$ and $\mathbf{1}_{--}$ of Z_2.

In addition to the above, there is another choice of $Z_2 \times Z_2$ subgroup, which consists of $\{e, a^2, ab, a^3b\}$. In this case, we can obtain the same decomposition of D_4.

• $D_4 \to Z_2$

Furthermore, both Z_4 and $Z_2 \times Z_2$ include Z_2 subgroup. The decomposition of D_4 to Z_2 is rather straightforward.

13.6 general D_N

Since the group D_N is isomorphic to $Z_N \rtimes Z_2$, D_M and Z_N as well as Z_2 appear as subgroups of D_N, where M is a divisor of N. Recall that all of D_N elements are written by $a^m b^k$ with $m = 0, \cdots, N-1$ and $k = 0, 1$. There are singlets and doublets $\mathbf{2}_k$, where $k = 1, \cdots, N/2 - 1$ for $N =$ even and $k = 1, \cdots, (N-1)/2$ for $N =$ odd. On the doublet $\mathbf{2}_k$, the generators a and b are represented as

$$a = \begin{pmatrix} \rho^k & 0 \\ 0 & \rho^{-k} \end{pmatrix}, \quad b = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \tag{385}$$

where $\rho = e^{2\pi i/N}$. For $N =$ even, there are four singlets $\mathbf{1}_{\pm\pm}$. The generator b is represented as $b = 1$ on $\mathbf{1}_{+\pm}$, while $b = -1$ on $\mathbf{1}_{-\pm}$. The generator a is represented as $a = 1$ on $\mathbf{1}_{++}$ and $\mathbf{1}_{--}$, while $a = -1$ on $\mathbf{1}_{+-}$ and $\mathbf{1}_{+-}$. For $N =$ odd, there are two singlets $\mathbf{1}_{\pm}$. The generator b is represented as $b = 1$ on $\mathbf{1}_+$ and $b = -1$ on $\mathbf{1}_-$, while $a = 1$ on both singlets.

• $D_N \to Z_2$

The two elements e and b construct the Z_2 subgroup. Obviously, there are two singlet representations $\mathbf{1}_0, \mathbf{1}_1$, where the subscript denotes the Z_2 charge. That is, we have $b = 1$ on $\mathbf{1}_0$ and $b = -1$ on $\mathbf{1}_1$.

When N is even, the singlets $\mathbf{1}_{++}$ and $\mathbf{1}_{+-}$ of D_N become $\mathbf{1}_0$ of Z_2 and the singlets $\mathbf{1}_{-+}$ and $\mathbf{1}_{--}$ of D_N become $\mathbf{1}_1$ of Z_2. The doublets $\mathbf{2}_k$ of D_N, (x_1, x_2), decompose two singlets as $\mathbf{1}_0 : x_1 + x_2$ and $\mathbf{1}_1 : x_1 - x_2$. These results are summarized as follows,

$$\begin{array}{cccccc} D_N & \mathbf{1}_{++} & \mathbf{1}_{+-} & \mathbf{1}_{-+} & \mathbf{1}_{--} & \mathbf{2}_k \\ & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\ Z_2 & \mathbf{1}_0 & \mathbf{1}_0 & \mathbf{1}_1 & \mathbf{1}_1 & \mathbf{1}_0 + \mathbf{1}_1 \end{array} . \tag{386}$$

When N is odd, the singlet $\mathbf{1}_+$ of D_N becomes $\mathbf{1}_0$ of Z_2 and the singlet $\mathbf{1}_-$ of D_N becomes $\mathbf{1}_1$ of Z_2. The decompositions of doublets $\mathbf{2}_k$ are the same as those for $N =$ even. These results are summarized as

$$\begin{array}{cccc} D_N & \mathbf{1}_{+} & \mathbf{1}_{-} & \mathbf{2}_k \\ & \downarrow & \downarrow & \downarrow \\ Z_2 & \mathbf{1}_0 & \mathbf{1}_1 & \mathbf{1}_0 + \mathbf{1}_1 \end{array} . \tag{387}$$

• $D_N \to Z_N$

The subgroup Z_N consists of the elements $\{e, a, \cdots, a^{N-1}\}$. Obviously it is the normal subgroups of D_N and there are N types of irreducible singlet representations $\mathbf{1}_0, \mathbf{1}_1, \cdots, \mathbf{1}_{N-1}$. On the $\mathbf{1}_k$, the generator a is represented as $a = \rho^k$.

When N is even, the singlets $\mathbf{1}_{++}$ and $\mathbf{1}_{--}$ of D_N become $\mathbf{1}_0$ of Z_N and the singlets $\mathbf{1}_{+-}$ and $\mathbf{1}_{-+}$ of D_N become $\mathbf{1}_{N/2}$ of Z_N. The doublets $\mathbf{2}_k$, (x_1, x_2), decompose to two singlets as $\mathbf{1}_k : x_1$ and $\mathbf{1}_{N-k} : x_2$. These results are summarized as follows,

$$\begin{array}{cccccc} D_N & \mathbf{1}_{++} & \mathbf{1}_{+-} & \mathbf{1}_{-+} & \mathbf{1}_{--} & \mathbf{2}_k \\ & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\ Z_N & \mathbf{1}_0 & \mathbf{1}_{N/2} & \mathbf{1}_{N/2} & \mathbf{1}_0 & \mathbf{1}_k + \mathbf{1}_{N-k} \end{array} . \tag{388}$$

When N is odd, both $\mathbf{1}_+$ and $\mathbf{1}_-$ of D_N become $\mathbf{1}_0$ of Z_N. The decompositions of doublets $\mathbf{2}_k$ are the same as those for $N =$ even.

• $D_N \to D_M$

The above decompositions of D_N are rather straightforward, because subgroups are Abelian. Here we consider the D_M subgroup, where M is a divisor of N. The decompositions of D_N to D_M would be non-trivial. We denote $\tilde{a} = a^\ell$ with $\ell = N/M$, where ℓ is integer. The subgroup D_M consists of $\tilde{a}^m b^k$ with $m = 0, \cdots, M-1$ and $k = 0, 1$. There are three combinations of (N, M), i.e. (N, M)=(even,even), (even,odd) and (odd,odd).

We start with the combination (N, M) =(even,even). Recall that (ab) of D_N is represented as $ab = 1$ on $\mathbf{1}_{\pm +}$ and $ab = -1$ on $\mathbf{1}_{\pm -}$. Thus, the representations of $(a^\ell b)$ depend on whether ℓ is even or odd. When ℓ is odd, (ab) and $(a^\ell b)$ are represented in the same way on each of the above singlets. On the other hand, when $\ell =$ even, we always have the singlet representations with $a^\ell = 1$. The doublets $\mathbf{2}_k$ of D_N correspond to the doublets $\mathbf{2}_{k'}$ of D_M when $k = k'$ (mod M). In addition, when $k = -k'$ (mod M), doublets $\mathbf{2}_k$ (x_1, x_2) of D_N correspond to the doublets $\mathbf{2}_{M-k'}$ (x_2, x_1) of D_M. That is, the components are exchanged each other and we denote it by $\tilde{\mathbf{2}}_{M-k'}$. Furthermore, the other doublets $\mathbf{2}_k$ of D_N decompose to two singlets of D_M as $\mathbf{1}_{+-}+\mathbf{1}_{-+}$ with $\mathbf{1}_{+-}: x_1+x_2$ and $\mathbf{1}_{-+}: x_1-x_2$ for $k=(M/2)$ (mod M) and $\mathbf{1}_{++}+\mathbf{1}_{--}$ with $\mathbf{1}_{++}: x_1+x_2$ and $\mathbf{1}_{--}: x_1-x_2$ for $k=0$ (mod M). These results are summarized as follows,

$$\begin{array}{cccccccccc} & D_N & \mathbf{1}_{++} & \mathbf{1}_{+-} & \mathbf{1}_{-+} & \mathbf{1}_{--} & \mathbf{2}_{k'+Mn} & \mathbf{2}_{\frac{M}{2}-k'+Mn} & \mathbf{2}_{\frac{M}{2}(2n+1)} & \mathbf{2}_{Mn} \\ & & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\ (\ell=\mathrm{odd}) & D_M & \mathbf{1}_{++} & \mathbf{1}_{+-} & \mathbf{1}_{-+} & \mathbf{1}_{--} & \mathbf{2}_{k'} & \tilde{\mathbf{2}}_{\frac{M}{2}-k'} & \mathbf{1}_{+-}+\mathbf{1}_{-+} & \mathbf{1}_{++}+\mathbf{1}_{--} \\ (\ell=\mathrm{even}) & D_M & \mathbf{1}_{++} & \mathbf{1}_{++} & \mathbf{1}_{--} & \mathbf{1}_{--} & \mathbf{2}_{k'} & \tilde{\mathbf{2}}_{\frac{M}{2}-k'} & \mathbf{1}_{+-}+\mathbf{1}_{-+} & \mathbf{1}_{++}+\mathbf{1}_{--} \end{array}, \tag{389}$$

where n is integer.

Next we consider the case with $(N,M)=$(even,odd). In this case, the singlet $\mathbf{1}_{++}$, $\mathbf{1}_{+-}$, $\mathbf{1}_{-+}$ and $\mathbf{1}_{--}$ of D_N become $\mathbf{1}_+$, $\mathbf{1}_+$, $\mathbf{1}_-$ and $\mathbf{1}_-$ of D_M, respectively. The doublets $\mathbf{2}_k$ of D_N correspond to the doublets $\mathbf{2}_{k'}$ of D_M when $k=k'$ (mod M). In addition, when $k=-k'$ (mod M), the doublets $\mathbf{2}_k$ (x_1,x_2) of D_N correspond to the doublets $\mathbf{2}_{M-k'}$ (x_2,x_1) of D_M. Furthermore, when $k=0$ (mod M), other doublets $\mathbf{2}_k$ of D_N decompose to two singlets of D_M as $\mathbf{1}_++\mathbf{1}_-$, where $\mathbf{1}_+: x_1+x_2$ and $\mathbf{1}_-: x_1-x_2$. These results are summarized as follows,

$$\begin{array}{cccccccc} D_N & \mathbf{1}_{++} & \mathbf{1}_{-+} & \mathbf{1}_{+-} & \mathbf{1}_{--} & \mathbf{2}_{k'+Mn} & \mathbf{2}_{Mn-k'} & \mathbf{2}_{Mn} \\ & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\ D_M & \mathbf{1}_+ & \mathbf{1}_+ & \mathbf{1}_- & \mathbf{1}_- & \mathbf{2}_{k'} & \tilde{\mathbf{2}}_{M-k'} & \mathbf{1}_++\mathbf{1}_- \end{array}, \tag{390}$$

where n is integer.

Now, let us consider the case with $(N,M)=$(odd,odd). In this case, the singlets $\mathbf{1}_+$ and $\mathbf{1}_-$ of D_N become $\mathbf{1}_+$ and $\mathbf{1}_-$ of D_M. The doublets $\mathbf{2}_k$ of D_N correspond to the doublets $\mathbf{2}_{k'}$ of D_M when $k=k'$ (mod M). In addition, when $k=-k'$ (mod M), doublets $\mathbf{2}_k$ (x_1,x_2) of D_N correspond to the doublets $\mathbf{2}_{M-k'}$ (x_2,x_1) of D_M. Furthermore, when $k=0$ (mod M), other doublets $\mathbf{2}_k$ of D_N decompose to two singlets of D_M as $\mathbf{1}_++\mathbf{1}_-$, where $\mathbf{1}_+: x_1+x_2$ and $\mathbf{1}_-: x_1-x_2$. These results are summarized as follows,

$$\begin{array}{cccccc} D_N & \mathbf{1}_+ & \mathbf{1}_- & \mathbf{2}_{k'+Mn} & \mathbf{2}_{Mn-k'} & \mathbf{2}_{Mn} \\ & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\ D_M & \mathbf{1}_+ & \mathbf{1}_- & \mathbf{2}_{k'} & \tilde{\mathbf{2}}_{M-k'} & \mathbf{1}_++\mathbf{1}_- \end{array}, \tag{391}$$

where n is integer.

13.7 Q_4

Here, we study Q_4. All of the Q_4 elements are written by $a^m b^k$ with $m=0,1,2,3$ and $k=0,1$. Since the order of Q_4 is equal to 8, it contains the order 2 and 4 subgroups. There are a few types of the order 4 groups which correspond to Z_4 groups.

- $Q_4 \to Z_4$

$(N=4n)$	$\mathbf{1}_{++}$	$\mathbf{1}_{+-}$	$\mathbf{1}_{-+}$	$\mathbf{1}_{--}$	$\mathbf{2}_{k=\text{odd}}$	$\mathbf{2}_{k=\text{even}}$
a	1	-1	-1	1	$\begin{pmatrix} \rho^k & 0 \\ 0 & \rho^{-k} \end{pmatrix}$	$\begin{pmatrix} \rho^k & 0 \\ 0 & \rho^{-k} \end{pmatrix}$
b	1	1	-1	-1	$\begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

Table 33: Representations of Q_N for $N = 4n$

For example, the elements $\{e, a, a^2, a^3\}$ construct the Z_4 subgroup. Obviously it is the normal subgroups of Q_4 and there are four types of irreducible singlet representations $\mathbf{1}_m$ with $m = 0, 1, 2, 3$, where a is represented as $a = e^{\pi i m/2}$. From the characters of Q_4 groups, it is found that $\mathbf{1}_{++}$ and $\mathbf{1}_{--}$ of Q_4 correspond to $\mathbf{1}_0$ of Z_4 and $\mathbf{1}_{-+}$ and $\mathbf{1}_{+-}$ of Q_4 correspond to $\mathbf{1}_2$ of Z_4. For the doublets of Q_4, it is convenient to use the diagonal base of matrix a as

$$a = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}. \tag{392}$$

Then we find that the doublet $\mathbf{2}$ (x_1, x_2) decomposes to two singlets as $\mathbf{1}_1 : x_1$ and $\mathbf{1}_3 : x_2$. These results are summarized as follows,

$$\begin{array}{cccccc} Q_4 & \mathbf{1}_{++} & \mathbf{1}_{+-} & \mathbf{1}_{-+} & \mathbf{1}_{--} & \mathbf{2} \\ & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\ Z_4 & \mathbf{1}_0 & \mathbf{1}_2 & \mathbf{1}_2 & \mathbf{1}_2 & \mathbf{1}_1 + \mathbf{1}_3 \end{array}. \tag{393}$$

In addition, other Z_4 subgroups consist of $\{e, b, b^2, b^3\}$ and $\{e, ab, (ab)^2, (ab)^3\}$. For those Z_4 subgroups, we obtain the same results when we choose proper basis. Furthermore, subgroups of Z_2 can appear from the above Z_4 groups. The decomposition of Z_4 to Z_2 is rather straightforward.

13.8 general Q_N

Recall that all of the Q_N elements are written as $a^m b^k$ with $m = 0, \cdots, N-1$ and $k = 0, 1$, where $a^N = e$ and $b^2 = a^{N/2}$. Similarly to D_N with $N =$ even, there are four singlets $\mathbf{1}_{\pm\pm}$ and doublets $\mathbf{2}_k$ with $k = 1, \cdots, N/2 - 1$. Tables 33 and 34 show the representations of a and b on these representations for $N = 4n$ and $N = 4n + 2$.

• $Q_N \to Z_4$

First, we consider the subgroup Z_4, which consists of the elements $\{e, b, b^2, b^3\}$. Obviously, there are four singlet representations $\mathbf{1}_m$ for Z_4 and the generator b is represented as $b = e^{\pi i m/2}$ on $\mathbf{1}_m$.

When $N = 4n$, $\mathbf{1}_{++}$ and $\mathbf{1}_{+-}$ of Q_N correspond to $\mathbf{1}_0$ of Z_4 and $\mathbf{1}_{-+}$ and $\mathbf{1}_{--}$ of Q_N correspond to $\mathbf{1}_2$ of Z_4. The doublets $\mathbf{2}_k$ of Q_N, (x_1, x_2) decompose to two singlets as

$(N=4n+2)$	$\mathbf{1}_{++}$	$\mathbf{1}_{+-}$	$\mathbf{1}_{-+}$	$\mathbf{1}_{--}$	$\mathbf{2}_{k=\text{odd}}$	$\mathbf{2}_{k=\text{even}}$
a	1	-1	-1	1	$\begin{pmatrix} \rho^k & 0 \\ 0 & \rho^{-k} \end{pmatrix}$	$\begin{pmatrix} \rho^k & 0 \\ 0 & \rho^{-k} \end{pmatrix}$
b	1	i	$-i$	-1	$\begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

Table 34: Representations of Q_N for $N=4n+2$

$\mathbf{1}_1 : (x_1 - ix_2)$ and $\mathbf{1}_3 : (x_1 + ix_2)$. These results are summarized as follows,

$$\begin{array}{cccccc} Q_N & \mathbf{1}_{++} & \mathbf{1}_{+-} & \mathbf{1}_{-+} & \mathbf{1}_{--} & \mathbf{2}_k \\ & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\ Z_4 & \mathbf{1}_0 & \mathbf{1}_0 & \mathbf{1}_2 & \mathbf{1}_2 & \mathbf{1}_1 + \mathbf{1}_3 \end{array} . \tag{394}$$

When $N=4n+2$, $\mathbf{1}_{++}$, $\mathbf{1}_{+-}$, $\mathbf{1}_{-+}$ and $\mathbf{1}_{--}$ of Q_N correspond to $\mathbf{1}_0$, $\mathbf{1}_1$, $\mathbf{1}_2$ and $\mathbf{1}_3$ of Z_4, respectively. The decompositions of doublets $\mathbf{2}_k$ are the same as those for $N=4n$. Then, these results are summarized as follows,

$$\begin{array}{cccccc} Q_N & \mathbf{1}_{++} & \mathbf{1}_{+-} & \mathbf{1}_{-+} & \mathbf{1}_{--} & \mathbf{2}_k \\ & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\ Z_4 & \mathbf{1}_0 & \mathbf{1}_1 & \mathbf{1}_2 & \mathbf{1}_3 & \mathbf{1}_1 + \mathbf{1}_3 \end{array} . \tag{395}$$

• $Q_N \to Z_N$

Next, we consider the subgroup Z_N, which consist of the elements $\{e, a, \cdots, a^{N-1}\}$. Obviously it is the normal subgroups of Q_N and there are N types of irreducible singlet representations $\mathbf{1}_0, \mathbf{1}_1, \cdots, \mathbf{1}_{N-1}$. On the singlet $\mathbf{1}_m$ of Z_N, the generator a is represented as $a = \rho^m$. The singlets, $\mathbf{1}_{++}$ and $\mathbf{1}_{--}$ of Q_N correspond to $\mathbf{1}_0$ of Z_N and the singlets $\mathbf{1}_{+-}$ and $\mathbf{1}_{-+}$ of Q_N correspond to $\mathbf{1}_{N/2}$ of Z_N. The doublets $\mathbf{2}_k$, (x_1, x_2), of Q_N decompose to two singlets as $\mathbf{1}_k : x_1$ and $\mathbf{1}_{N-k} : x_2$. These results are summarizes as follows,

$$\begin{array}{cccccc} Q_N & \mathbf{1}_{++} & \mathbf{1}_{+-} & \mathbf{1}_{-+} & \mathbf{1}_{--} & \mathbf{2}_k \\ & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\ Z_2 & \mathbf{1}_0 & \mathbf{1}_{N/2} & \mathbf{1}_{N/2} & \mathbf{1}_0 & \mathbf{1}_k + \mathbf{1}_{N-k} \end{array} . \tag{396}$$

• $Q_N \to Q_M$

We consider the Q_M subgroup, where M is a divisor of N. We denote $\tilde{a} = a^\ell$ with $\ell = N/M$, where $\ell =$ integer. The subgroup Q_M consists of $\tilde{a}^m b^k$ with $m = 0, \cdots, M-1$ and $k = 0, 1$. There are three combinations of (N, M), i.e. $(N, M) = (4n, 4m)$, $(4n, 4m+2)$ and $(4n+2, 4m+2)$.

We start with the combination $(N, M) = (4n, 4m)$, where $\ell = N/M$ can be even or odd. Recall that (ab) of Q_N is represented as $ab = 1$ on $\mathbf{1}_{\pm +}$ and $ab = -1$ on $\mathbf{1}_{\pm -}$. Thus, the representations of $(a^\ell b)$ depend on whether ℓ is even or odd. When ℓ is odd, (ab) and

$(a^\ell b)$ are represented in the same way as on each of the above singlets. On the other hand, when ℓ = even, we always have the singlet representations with $a^\ell = 1$. The doublets $\mathbf{2}_k$ of Q_N correspond to the doublets $\mathbf{2}_{k'}$ of Q_M when $k = k'$ (mod M). In addition, when $k = -k'$ (mod M), doublets $\mathbf{2}_k$ (x_1, x_2) of Q_N correspond to the doublets $\mathbf{2}_{M-k'}$ (x_2, x_1) of Q_M. Furthermore, other doublets $\mathbf{2}_k$ of Q_N decompose to two singlets of Q_M as $\mathbf{1}_{+-}+\mathbf{1}_{-+}$ with $\mathbf{1}_{+-} : x_1+x_2$ and $\mathbf{1}_{-+} : x_1-x_2$ for $k = (M/2)$ (mod M) and $\mathbf{1}_{++}+\mathbf{1}_{--}$ with $\mathbf{1}_{++} : x_1 + x_2$ and $\mathbf{1}_{--} : x_1 - x_2$ for $k = 0$ (mod M). These results are summarized as follows,

$$\begin{array}{cccccccccc} (N=4n) & Q_N & \mathbf{1}_{++} & \mathbf{1}_{+-} & \mathbf{1}_{-+} & \mathbf{1}_{--} & \mathbf{2}_{k+Mk'} & \mathbf{2}_{Mk'-k} & \mathbf{2}_{\frac{M}{2}(2k'+1)} & \mathbf{2}_{Mk'} \\ (M=4m) & & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\ (\ell=\text{odd}) & Q_M & \mathbf{1}_{++} & \mathbf{1}_{+-} & \mathbf{1}_{-+} & \mathbf{1}_{--} & \mathbf{2}_k & \tilde{\mathbf{2}}_{M-k} & \mathbf{1}_{+-}+\mathbf{1}_{-+} & \mathbf{1}_{++}+\mathbf{1}_{--} \\ (\ell=\text{even}) & Q_M & \mathbf{1}_{++} & \mathbf{1}_{++} & \mathbf{1}_{--} & \mathbf{1}_{--} & \mathbf{2}_k & \tilde{\mathbf{2}}_{M-k} & \mathbf{1}_{+-}+\mathbf{1}_{-+} & \mathbf{1}_{++}+\mathbf{1}_{--} \end{array}, \tag{397}$$

where k' is integer.

Next we consider the case with $(N, M) = (4n, 4m+2)$, where ℓ must be even. Similarly to the above case with ℓ = even, the singlets $\mathbf{1}_{++}$, $\mathbf{1}_{+-}$, $\mathbf{1}_{-+}$ and $\mathbf{1}_{--}$ of Q_N correspond to $\mathbf{1}_{++}$, $\mathbf{1}_{++}$, $\mathbf{1}_{--}$ and $\mathbf{1}_{--}$ of Q_M. The results on decompositions of doublets are also the same as the above case with $(N, M) = (4n, 4m)$ and $\ell = N/M$ = even. These results are summarized as follows,

$$\begin{array}{cccccccccc} (N=4n) & Q_N & \mathbf{1}_{++} & \mathbf{1}_{+-} & \mathbf{1}_{-+} & \mathbf{1}_{--} & \mathbf{2}_{k+Mk'} & \mathbf{2}_{Mk'-k} & \mathbf{2}_{\frac{M}{2}(2k'+1)} & \mathbf{2}_{Mk'} \\ (M=4m+2) & & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\ (\ell=\text{even}) & Q_M & \mathbf{1}_{++} & \mathbf{1}_{++} & \mathbf{1}_{--} & \mathbf{1}_{--} & \mathbf{2}_k & \tilde{\mathbf{2}}_{M-k} & \mathbf{1}_{+-}+\mathbf{1}_{-+} & \mathbf{1}_{++}+\mathbf{1}_{--} \end{array}, \tag{398}$$

where k' is integer.

Next, we consider the case with $(N, M) = (4n + 2, 4m + 2)$, where ℓ must be odd. In this case, the results on decompositions are the same as the case with $(N, M) = (4n, 4m)$ and $\ell = N/M$ = odd. These results are summarized as follows,

$$\begin{array}{cccccccccc} (N=4n+2) & Q_N & \mathbf{1}_{++} & \mathbf{1}_{-+} & \mathbf{1}_{+-} & \mathbf{1}_{--} & \mathbf{2}_{k+Mk'} & \mathbf{2}_{Mk'-k} & \mathbf{2}_{\frac{M}{2}(2k'+1)} & \mathbf{2}_{Mk'} \\ (M=4m+2) & & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\ (\ell=\text{odd}) & Q_M & \mathbf{1}_{++} & \mathbf{1}_{-+} & \mathbf{1}_{+-} & \mathbf{1}_{--} & \mathbf{2}_k & \tilde{\mathbf{2}}_{M-k} & \mathbf{1}_{+-}+\mathbf{1}_{-+} & \mathbf{1}_{++}+\mathbf{1}_{--} \end{array}, \tag{399}$$

where k' = integer.

13.9 general $\Sigma(2N^2)$

Recall that all of the $\Sigma(2N^2)$ elements are written by $b^k a^m a'^n$ with $k = 0, 1$ and $m, n = 0, 1, \cdots, N-1$. The generators, a, a' and b, satisfy $a^N = a'^N = b^2 = e$, $aa' = a'a$ and $bab = a'$, that is, a, a' and b correspond to Z_N, Z'_N and Z_2 of $(Z_N \times Z'_N) \rtimes Z_2$, respectively. Table 35 shows the representations of these generators on each representation. The number of doublets $\mathbf{2}_{p,q}$ is equal to $N(N-1)/2$ with the relation $p > q$.

	$\mathbf{1}_{+n}$	$\mathbf{1}_{-n}$	$\mathbf{2}_{p,q}$
a	ρ^n	ρ^n	$\begin{pmatrix} \rho^q & 0 \\ 0 & \rho^p \end{pmatrix}$
a'	ρ^n	ρ^n	$\begin{pmatrix} \rho^p & 0 \\ 0 & \rho^q \end{pmatrix}$
b	1	-1	$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

Table 35: Representations of $\Sigma(2N^2)$

• $\Sigma(2N^2) \to Z_2$

The subgroup Z_2 consists of the elements $\{e, b\}$. There are two singlet representations $\mathbf{1}_0, \mathbf{1}_1$ for Z_2 and the generator b is represented as $b = (-1)^m$ on $\mathbf{1}_m$. Then, each representation of $\Sigma(2N^2)$ is decomposed as

$$\begin{array}{cccc} \Sigma(2N^2) & \mathbf{1}_{+n} & \mathbf{1}_{-n} & \mathbf{2}_{\ell,m} \\ & \downarrow & \downarrow & \downarrow \\ Z_2 & \mathbf{1}_0 & \mathbf{1}_1 & \mathbf{1}_0 + \mathbf{1}_1 \end{array}, \tag{400}$$

where the components of doublets (x_1, x_2) correspond to $\mathbf{1}_0 : (x_1 + x_2)$ and $\mathbf{1}_1 : (x_1 - x_2)$.

• $\Sigma(2N^2) \to Z_N \times Z_N$

The subgroup $Z_N \times Z_N$ consists of the elements $a^m a'^n$ with $m, n = 0, \cdots, N-1$. Obviously it is the normal subgroups of $\Sigma(2N^2)$. There are N^2 singlet representations $\mathbf{1}_{m,n}$ and the generators a and a' are represented as $a = \rho^m$ and $a' = \rho^n$ on $\mathbf{1}_{m,n}$. Then, each representation of $\Sigma(2N^2)$ is decomposed as

$$\begin{array}{cccc} \Sigma(2N^2) & \mathbf{1}_{+n} & \mathbf{1}_{-n} & \mathbf{2}_{\ell,m} \\ & \downarrow & \downarrow & \downarrow \\ Z_N \times Z_N & \mathbf{1}_{n,n} & \mathbf{1}_{n,n} & \mathbf{1}_{\ell,m} + \mathbf{1}_{m,\ell} \end{array}. \tag{401}$$

• $\Sigma(2N^2) \to D_N$

We consider D_N as a subgroup of $\Sigma(2N^2)$. We denote $\tilde{a} = a^{-1}a'$. Then, the subgroup D_N consists of the elements $\tilde{a}^m b^k$ with $k = 0, 1$ and $m = 0, \cdots, N-1$. Table 36 shows the representations of the generators, $\tilde{a}$ and b, on each representation of $\Sigma(2N^2)$.

At first, we consider the case that N is even. The doublets $\mathbf{2}_{p,q}$ of $\Sigma(2N^2)$ are still doublets of D_N except $p - q = \frac{N}{2}$. On the other hand, when $p - q = \frac{N}{2}$, the doublets decompose to two singlets of D_N. Then, each representation of $\Sigma(2N^2)$ is decomposed as

$$\begin{array}{cccccc} \Sigma(2N^2) & \mathbf{1}_{+n} & \mathbf{1}_{-n} & \mathbf{2}_{q+k',q} & \mathbf{2}_{q-k',q} & \mathbf{2}_{q+\frac{N}{2},q} \\ & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\ D_N & \mathbf{1}_{++} & \mathbf{1}_{--} & \mathbf{2}_{k'} & \tilde{\mathbf{2}}_{k'} & \mathbf{1}_{+-} + \mathbf{1}_{-+} \end{array}. \tag{402}$$

	$\mathbf{1}_{+n}$	$\mathbf{1}_{-n}$	$\mathbf{2}_{p,q}$
$\tilde{a}$	1	1	$\begin{pmatrix} \rho^{p-q} & 0 \\ 0 & \rho^{-(p-q)} \end{pmatrix}$
b	1	-1	$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

Table 36: Representations of $\tilde{a}$ and b in $\Sigma(2N^2)$

	$\mathbf{1}_{+n}$	$\mathbf{1}_{-n}$	$\mathbf{2}_{p,q}$
$\tilde{a}$	1	1	$\begin{pmatrix} \rho^{p-q} & 0 \\ 0 & \rho^{-(p-q)} \end{pmatrix}$
$\tilde{b}$	1	-1	$\begin{pmatrix} 0 & (-1)^q \\ (-1)^p & 0 \end{pmatrix}$

Table 37: Representations of $\tilde{a}$ and $\tilde{b}$ in $\Sigma(2N^2)$

Next, we consider the case that N is odd. In this case, each representation of $\Sigma(2N^2)$ is decomposed as

$$\begin{array}{ccccc} \Sigma(2N^2) & \mathbf{1}_{+n} & \mathbf{1}_{-n} & \mathbf{2}_{q+k',q} & \mathbf{2}_{q-k',q} \\ & \downarrow & \downarrow & \downarrow & \downarrow \\ D_N & \mathbf{1}_{+} & \mathbf{1}_{-} & \mathbf{2}_{k'} & \tilde{\mathbf{2}}_{N-k'} \end{array} . \tag{403}$$

- $\Sigma(2N^2) \to Q_N$

We consider Q_N as a subgroup of $\Sigma(2N^2)$ with $N =$ even. We denote $\tilde{a} = a^{-1}a'$ and $\tilde{b} = ba'^{N/2}$. Then, the subgroup Q_N consists of $\tilde{a}^m\tilde{b}^k$ with $m = 0, \cdots, N-1$ and $k = 0, 1$. Table 37 shows the representations of these generators $\tilde{a}$ and $\tilde{b}$ on each representation of $\Sigma(2N^2)$. Then the singlets of $\Sigma(2N^2)$ become singlets of Q_N as follows

$$\begin{array}{cccc} \Sigma(2N^2) & \mathbf{1}_{+n} & \mathbf{1}_{-n} & \\ & \downarrow & \downarrow & \\ Q_N & \mathbf{1}_{++}, & \mathbf{1}_{--}, & (n:\text{even}) \\ & \mathbf{1}_{--}, & \mathbf{1}_{++}, & (n:\text{odd}) \end{array} . \tag{404}$$

The decompositions of doublets are obtained in a way similar to the decomposition, $\Sigma(2N^2) \to D_N$, as follows,

$$\begin{array}{ccc} \Sigma(2N^2) & \mathbf{2}_{q+k',q} & \mathbf{2}_{q+\frac{N}{2}} \\ & \downarrow & \downarrow \\ Q_N & \mathbf{2}_{k'} & \mathbf{1}_{+-} + \mathbf{1}_{-+} \end{array} . \tag{405}$$

- $\Sigma(2N^2) \to \Sigma(2M^2)$

	$\mathbf{1}_{+n}$	$\mathbf{1}_{-n}$	$\mathbf{2}_{p,q}$
$\tilde{a}$	$\rho^{n\ell}$	$\rho^{n\ell}$	$\begin{pmatrix} \rho^{q\ell} & 0 \\ 0 & \rho^{p\ell} \end{pmatrix}$
$\tilde{a}'$	$\rho^{n\ell}$	$\rho^{n\ell}$	$\begin{pmatrix} \rho^{p\ell} & 0 \\ 0 & \rho^{q\ell} \end{pmatrix}$
b	1	-1	$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

Table 38: Representations of $\tilde{a}$, $\tilde{a}'$ and $\tilde{b}$ in $\Sigma(2N^2)$

We consider the subgroup $\Sigma(2M^2)$, where M is a divisor of N. We denote $\tilde{a} = a^\ell$ and $\tilde{a}' = a'^\ell$ with $\ell = N/M$, where $\ell =$ integer. The subgroup $\Sigma(2M^2)$ consists of $b^k\tilde{a}^m\tilde{a}'^n$ with $k = 0, 1$ and $m, n = 0, \cdots, M-1$. Table 38 shows the representations of $\tilde{a}$, $\tilde{a}'$ and $\tilde{b}$ on each representation of $\Sigma(2N^2)$. Then, representations of $\Sigma(2N^2)$ correspond to representations of $\Sigma(2M^2)$ as follows,

$$\begin{array}{cccc} \Sigma(2N^2) & \mathbf{1}_{+n} & \mathbf{1}_{-n} & \mathbf{2}_{p+Mn,q+Mn'} \\ & \downarrow & \downarrow & \downarrow \\ \Sigma(2m^2) & \mathbf{1}_{+n} & \mathbf{1}_{-n} & \mathbf{2}_{p,q} \end{array}, \tag{406}$$

where n, n' are integers.

13.10 $\Sigma(32)$

The $\Sigma(32)$ group includes subgroups, D_4, Q_4 and $\Sigma(8)$ as well as Abelian groups, as shown in the previous section. In addition, the $\Sigma(32)$ group includes the subgroup, which has not been studied in the previous sections. It is useful to construct a discrete group as a subgroup of known groups, as explained in section 10, where the example T_7 was shown. Here, we show another example $(Z_4 \times Z_2) \rtimes Z_2$ as a subgroup of $\Sigma(32) \simeq (Z_4 \times Z_4) \rtimes Z_2$.

All of the $\Sigma(32)$ elements are written by $b^k a^m a'^n$ with $k = 0, 1$ and $m, n = 0, 1, 2, 3$. The generators, a, a' and b, satisfy $a^4 = a'^4 = b^2 = e$, $aa' = a'a$ and $bab = a'$. Here we define $\tilde{a} = aa'$ and $\tilde{a}' = a^2$, where $\tilde{a}^4 = e$ and $\tilde{a}'^2 = e$. Then, the elements $b^k\tilde{a}^m\tilde{a}'^n$ with $k, n = 0, 1$ and $m = 0, 1, 2, 3$ construct a closed subalgebra, i.e. $(Z_4 \times Z_2) \rtimes Z_2$. It has ten conjugacy classes and eight singlets, $\mathbf{1}_{\pm 0}$, $\mathbf{1}_{\pm 1}$, $\mathbf{1}_{\pm 2}$ and $\mathbf{1}_{\pm 3}$, and two doublets, $\mathbf{2}_1$ and $\mathbf{2}_2$. These conjugacy classes and characters are shown in Table 39. From this table, we can find decompositions of $\Sigma(32)$ representations to representation of $(Z_4 \times Z_2) \rtimes Z_2$ as follows,

$$\begin{array}{cccccc} \Sigma(32) & \mathbf{1}_{\pm 0,\pm 1,\pm 2,\pm 3} & \mathbf{2}_{1,0}, \mathbf{2}_{3,2} & \mathbf{2}_{3,0}, \mathbf{2}_{2,1} & \mathbf{2}_{2,0} & \mathbf{2}_{3,1} \\ & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\ (Z_4 \times Z_2) \rtimes Z_2 & \mathbf{1}_{\pm 0,\pm 1,\pm 0,\pm 1} & \mathbf{2}_1 & \mathbf{2}_2 & \mathbf{1}_{+3} + \mathbf{1}_{-3} & \mathbf{1}_{+2} + \mathbf{1}_{-2}. \end{array} \tag{407}$$

		h	$\chi_{\pm 0}$	$\chi_{\pm 1}$	$\chi_{\pm 2}$	$\chi_{\pm 3}$	χ_{2_1}	χ_{2_2}
C_1:	$\{e\}$,	1	1	1	1	1	2	2
$C_1^{(1)}$:	$\{\tilde{a}\tilde{a}'\}$,	4	1	-1	1	-1	$2i$	$-2i$
$C_1^{(2)}$:	$\{\tilde{a}^2\tilde{a}'^2\}$,	2	1	1	1	1	-2	-2
$C_1^{(3)}$:	$\{\tilde{a}^3\tilde{a}'^3\}$,	4	1	-1	1	-1	$-2i$	$2i$
$C_2^{(0)}$:	$\{b, b\tilde{a}^2\tilde{a}'^2\}$,	2	± 1	± 1	± 1	± 1	0	0
$C_2^{(0)}$:	$\{b\tilde{a}\tilde{a}', b\tilde{a}^3\tilde{a}'^3\}$,	4	± 1	∓ 1	± 1	∓ 1	0	0
$C_2^{(0)}$:	$\{b\tilde{a}^2, b\tilde{a}'^2\}$,	4	± 1	∓ 1	∓ 1	± 1	0	0
$C_2^{(0)}$:	$\{b\tilde{a}\tilde{a}'^3, b\tilde{a}^3\tilde{a}'\}$,	2	± 1	± 1	∓ 1	∓ 1	0	0
$C_2^{(2,0)}$:	$\{\tilde{a}^2, \tilde{a}'^2\}$,	2	1	-1	-1	1	0	0
$C_2^{(3,1)}$:	$\{\tilde{a}\tilde{a}'^3, \tilde{a}^3\tilde{a}'\}$,	2	1	1	-1	-1	0	0

Table 39: Conjugacy classes and characters of $(Z_4 \times Z_2) \rtimes Z_2$

	$\mathbf{1}_k$	$\mathbf{3}_{[k][\ell]}$
a	1	$\begin{pmatrix} \rho^\ell & 0 & 0 \\ 0 & \rho^k & 0 \\ 0 & 0 & \rho^{-k-\ell} \end{pmatrix}$
a'	1	$\begin{pmatrix} \rho^{-k-\ell} & 0 & 0 \\ 0 & \rho^\ell & 0 \\ 0 & 0 & \rho^k \end{pmatrix}$
b	ω^k	$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$

Table 40: Representations of a, a' and b in $\Delta(3N^2)$ for $N/3 \neq$ integer

13.11 General $\Delta(3N^2)$

All of the $\Delta(3N^2)$ elements are written by $b^k a^m a'^n$ with $k = 0, 1, 2$ and $m, n = 0, \cdots, N-1$, where the generators, b, a and a', correspond to Z_3, Z_N and Z'_N of $(Z_N \times Z'_N) \rtimes Z_3$, respectively. Table 40 shows the representations of generators, b, a and a' on each representation of $\Delta(3N^2)$ for $N/3 \neq$ integer. Also Table 41 shows the same for $N/3 =$ integer.

• $\Delta(3N^2) \rightarrow Z_3$

The subgroup Z_3 consists of $\{e, b, b^2\}$. There are three singlet representations $\mathbf{1}_m$ with $m = 0, 1, 2$ for Z_3 and the generator b is represented as $b = \omega^m$ on $\mathbf{1}_m$. When $N/3 \neq$

	$\mathbf{1}_{k,\ell}$	$\mathbf{3}_{[k][\ell]}$
a	ω^ℓ	$\begin{pmatrix} \rho^\ell & 0 & 0 \\ 0 & \rho^k & 0 \\ 0 & 0 & \rho^{-k-\ell} \end{pmatrix}$
a'	ω^ℓ	$\begin{pmatrix} \rho^{-k-\ell} & 0 & 0 \\ 0 & \rho^\ell & 0 \\ 0 & 0 & \rho^k \end{pmatrix}$
b	ω^k	$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$

Table 41: Representations of a, a' and b in $\Delta(3N^2)$ for $N/3$ = integer

integer, each representation of $\Delta(3N^2)$ is decomposed as

$$\begin{array}{ccc} \Delta(3N^2) & \mathbf{1}_k & \mathbf{3}_{[k][\ell]} \\ & \downarrow & \downarrow \\ Z_3 & \mathbf{1}_k & \mathbf{1}_0+\mathbf{1}_1+\mathbf{1}_2 \end{array}. \tag{408}$$

On the other hand, when $N/3$ = integer, each representation of $\Delta(3N^2)$ is decomposed as

$$\begin{array}{ccc} \Delta(3N^2) & \mathbf{1}_{k,\ell} & \mathbf{3}_{[k][\ell]} \\ & \downarrow & \downarrow \\ Z_3 & \mathbf{1}_k & \mathbf{1}_0+\mathbf{1}_1+\mathbf{1}_2 \end{array}. \tag{409}$$

In both cases, the triplet components (x_1, x_2, x_3) of $\Delta(3N^2)$ are decomposed to singlets of Z_3 as $\mathbf{1}_0 : x_1 + x_2 + x_3$, $\mathbf{1}_1 : x_1 + \omega^2 x_2 + \omega x_3$ and $\mathbf{1}_2 : x_1 + \omega x_2 + \omega^2 x_3$.

- $\Delta(3N^2) \to Z_N \times Z_N$

The subgroup $Z_N \times Z_N$ consists of $\{a^m a'^n\}$ with $m, n = 0, 1, \cdots N-1$. There are N^2 singlet representations $\mathbf{1}_{m,n}$ and the generators a and a' are represented as $a = \rho^m$ and $a' = \rho^n$ on $\mathbf{1}_{m,n}$. When $N/3 \neq$ integer, each representation of $\Delta(3N^2)$ is decomposed as

$$\begin{array}{ccc} \Delta(3N^2) & \mathbf{1}_k & \mathbf{3}_{[k][\ell]} \\ & \downarrow & \downarrow \\ Z_N \times Z_N & \mathbf{1}_{0,0} & \mathbf{1}_{\ell,-k-\ell}+\mathbf{1}_{k,\ell}+\mathbf{1}_{-k-\ell,k} \end{array}. \tag{410}$$

In addition, when $N/3$ = integer, we have the following decompositions

$$\begin{array}{ccc} \Delta(3N^2) & \mathbf{1}_{k,\ell} & \mathbf{3}_{[k][\ell]} \\ & \downarrow & \downarrow \\ Z_N \times Z_N & \mathbf{1}_{N\ell/3,N\ell/3} & \mathbf{1}_{\ell,-k-\ell}+\mathbf{1}_{k,\ell}+\mathbf{1}_{-k-\ell,k} \end{array}. \tag{411}$$

- $\Delta(3N^2) \to \Delta(3M^2)$

	$\mathbf{1}_{k,\ell}$	$\mathbf{3}_{[k][\ell]}$
$\tilde{a}$	$\omega^{p\ell}$	$\begin{pmatrix} \rho^{p\ell} & 0 & 0 \\ 0 & \rho^{pk} & 0 \\ 0 & 0 & \rho^{-p(k+\ell)} \end{pmatrix}$
$\tilde{a}'$	$\omega^{p\ell}$	$\begin{pmatrix} \rho^{-p(k+\ell)} & 0 & 0 \\ 0 & \rho^{p\ell} & 0 \\ 0 & 0 & \rho^{pk} \end{pmatrix}$
b	ω^{k}	$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$

Table 42: Representations of $\tilde{a}$, $\tilde{a}'$ and $\tilde{b}$ in $\Delta(3N^2)$ for $N/3 =$ integer

	$\mathbf{1}_{k}$	$\mathbf{3}_{[k][\ell]}$
$\tilde{a}$	1	$\begin{pmatrix} \rho^{p\ell} & 0 & 0 \\ 0 & \rho^{pk} & 0 \\ 0 & 0 & \rho^{-p(k+\ell)} \end{pmatrix}$
$\tilde{a}'$	1	$\begin{pmatrix} \rho^{-p(k+\ell)} & 0 & 0 \\ 0 & \rho^{p\ell} & 0 \\ 0 & 0 & \rho^{pk} \end{pmatrix}$
b	ω^{k}	$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$

Table 43: Representations of $\tilde{a}$, $\tilde{a}'$ and $\tilde{b}$ in $\Delta(3N^2)$ for $N/3 \neq$ integer

We consider the subgroup $\Delta(3M^2)$, where M is a divisor of N. We denote $\tilde{a} = a^p$ and $\tilde{a}' = a'^p$ with $p = N/M$, where $p =$ integer. The subgroup $\Delta(3M^2)$ consists of $b^k \tilde{a}^m \tilde{a}'^n$ with $k = 0, 1, 2$ and $m, n = 0, \cdots, M-1$. Table 42 shows the representations of $\tilde{a}$, $\tilde{a}'$ and $\tilde{b}$ on each representation of $\Delta(3N^2)$ for $N/3 =$ integer. In addition, Table 43 shows the representations of $\tilde{a}$, $\tilde{a}'$ and $\tilde{b}$ on each representation of $\Delta(3N^2)$ for $N/3 \neq$ integer. There are three types of combinations (N, M), i.e. (1) both $N/3$ and $M/3$ are integers, (2) $N/3$ is integer, but $M/3$ is not integer, (3) either $N/3$ or $M/3$ is not integer.

When both $N/3$ and $M/3$ are integers, each representation of $\Delta(3N^2)$ is decomposed to representations of $\Delta(3M^2)$ as follows,

$$\begin{array}{ccc} \Delta(3N^2) & \mathbf{1}_{k,\ell} & \mathbf{3}_{[k+Mn][\ell+Mn']} \\ & \downarrow & \downarrow \\ \Delta(3M^2) & \mathbf{1}_{k,p\ell} & \mathbf{3}_{[k][\ell]} \end{array}, \tag{412}$$

where n and n' are integers.

Next we consider the case that $N/3 =$ integer and $M/3 \neq$ integer, where $p = N/M$

	$\mathbf{1}_k$	$\mathbf{3}$
a	1	$\begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$
a'	1	$\begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$
b	ω^k	$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$

Table 44: Representations of $\tilde{a}$, $\tilde{a}'$ and $\tilde{b}$ in $\Delta(12)$

must be $3n$. In this case, each representation of $\Delta(3N^2)$ is decomposed to representations of $\Delta(3M^2)$ as follows,

$$\begin{array}{ccc} \Delta(3N^2) & \mathbf{1}_{k,\ell} & \mathbf{3}_{[k+Mn][\ell+Mn']} \\ & \downarrow & \downarrow \\ \Delta(3M^2) & \mathbf{1}_k & \mathbf{3}_{[k][\ell]} \end{array}, \tag{413}$$

where n and n' are integers.

The last case is that either $N/3$ or $M/3$ is not integer. In this case, each representation of $\Delta(3N^2)$ is decomposed to representations of $\Delta(3M^2)$ as follows,

$$\begin{array}{ccc} \Delta(3N^2) & \mathbf{1}_k & \mathbf{3}_{[k+Mn][\ell+Mn']} \\ & \downarrow & \downarrow \\ \Delta(3M^2) & \mathbf{1}_k & \mathbf{3}_{[k][\ell]} \end{array}, \tag{414}$$

where n and n' are integers.

13.12 A_4

The A_4 group is isomorphic to $\Delta(12)$. Here, we apply the above generic results to the A_4 group. All of the $\Delta(12)$ elements are written by $b^k a^m a'^n$ with $k = 0, 1, 2$ and $m, n = 0, 1$. Table 44 shows the representations of generators a, a' and b on each representation.

- $A_4 \to Z_3$

The Z_3 group consists of $\{e, b, b^2\}$. Each representation of $\Delta(12)$ is decomposed as

$$\begin{array}{ccc} A_4 \simeq \Delta(12) & \mathbf{1}_k & \mathbf{3} \\ & \downarrow & \downarrow \\ Z_3 & \mathbf{1}_k & \mathbf{1}_0 + \mathbf{1}_1 + \mathbf{1}_2 \end{array}. \tag{415}$$

Decomposition of triplet (x_1, x_2, x_3) is obtained as $\mathbf{1}_0 : x_1 + x_2 + x_3$, $\mathbf{1}_1 : x_1 + \omega^2 x_2 + \omega x_3$ and $\mathbf{1}_2 : x_1 + \omega x_2 + \omega^2 x_3$.

	$\mathbf{1}_0$	$\mathbf{1}_1$	$\mathbf{1}_2$	$\mathbf{3}$	$\bar{\mathbf{3}}$
a	1	1	1	$\begin{pmatrix} \rho & 0 & 0 \\ 0 & \rho^2 & 0 \\ 0 & 0 & \rho^4 \end{pmatrix}$	$\begin{pmatrix} \rho^{-1} & 0 & 0 \\ 0 & \rho^{-2} & 0 \\ 0 & 0 & \rho^{-4} \end{pmatrix}$
b	1	ω	ω^2	$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$

Table 45: Representations of a and b in T_7

• $A_4 \to Z_2 \times Z_2$

The subgroup $Z_2 \times Z_2$ consists of $\{e, a, a', aa'\}$. Each representation of $\Delta(12)$ is decomposed as

$$\begin{array}{ccc} A_4 \simeq \Delta(12) & \mathbf{1}_k & \mathbf{3} \\ & \downarrow & \downarrow \\ Z_2 \times Z_2 & \mathbf{1}_{0,0} & \mathbf{1}_{1,1} + \mathbf{1}_{0,1} + \mathbf{1}_{1,0} \end{array} . \tag{416}$$

13.13 T_7

All of the T_7 elements are written as $b^m a^n$ with $m = 0, 1, 2$ and $n = 0, \cdots, 6$, where $b^3 = e$ and $a^7 = e$. Table 45 shows the representation of generators a and b on each representation of T_7.

• $T_7 \to Z_3$

The subgroup Z_3 consists of $\{e, b, b^2\}$. The three singlet representations $\mathbf{1}_m$ of Z_3 with $m = 0, 1, 2$ are specified such that $b = \omega^m$ on $\mathbf{1}_m$. Then, each representation of T_7 is decomposed as

$$\begin{array}{cccccc} T_7 & \mathbf{1}_0 & \mathbf{1}_1 & \mathbf{1}_2 & \mathbf{3} & \bar{\mathbf{3}} \\ & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\ Z_3 & \mathbf{1}_0 & \mathbf{1}_1 & \mathbf{1}_2 & \mathbf{1}_0 + \mathbf{1}_1 + \mathbf{1}_2 & \mathbf{1}_0 + \mathbf{1}_1 + \mathbf{1}_2 \end{array} . \tag{417}$$

Here the T_7 triplet $\mathbf{3} : (x_1, x_2, x_3)$ decomposes to three singlets, $\mathbf{1}_0 + \mathbf{1}_1 + \mathbf{1}_2$, and their components correspond to

$$\mathbf{1}_0 \ : \ x + y + z, \qquad \mathbf{1}_1 \ : \ x + \omega^2 y + \omega z, \qquad \mathbf{1}_2 \ : \ x + \omega y + \omega^2 z. \tag{418}$$

• $T_7 \to Z_7$

The subgroup Z_7 consists of a^n with $n = 0, \cdots, 6$. The seven singlets $\mathbf{1}_m$ of Z_7 with $m = 0, \cdots, 6$ are specified such that $b = \rho^m$ on $\mathbf{1}_m$. Then, each representation of T_7 is decomposed as

$$\begin{array}{cccccc} T_7 & \mathbf{1}_0 & \mathbf{1}_1 & \mathbf{1}_2 & \mathbf{3} & \bar{\mathbf{3}} \\ & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\ Z_3 & \mathbf{1}_0 & \mathbf{1}_0 & \mathbf{1}_0 & \mathbf{1}_1 + \mathbf{1}_2 + \mathbf{1}_4 & \mathbf{1}_3 + \mathbf{1}_5 + \mathbf{1}_6 \end{array} . \tag{419}$$

	$\mathbf{1}_\ell^k$	$\mathbf{3}_A$	$\mathbf{3}_B$	$\mathbf{3}_C$	$\mathbf{3}_D$
a	ω^k	$\begin{pmatrix} \omega & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$	$\begin{pmatrix} \omega^2 & 0 & 0 \\ 0 & \omega & 0 \\ 0 & 0 & \omega \end{pmatrix}$	$\begin{pmatrix} 1 & 0 & 0 \\ 0 & \omega^2 & 0 \\ 0 & 0 & \omega^2 \end{pmatrix}$	$\begin{pmatrix} \omega^2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \omega \end{pmatrix}$
a'	ω^k	$\begin{pmatrix} 1 & 0 & 0 \\ 0 & \omega & 0 \\ 0 & 0 & 1 \end{pmatrix}$	$\begin{pmatrix} \omega & 0 & 0 \\ 0 & \omega^2 & 0 \\ 0 & 0 & \omega \end{pmatrix}$	$\begin{pmatrix} \omega^2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \omega^2 \end{pmatrix}$	$\begin{pmatrix} \omega & 0 & 0 \\ 0 & \omega^2 & 0 \\ 0 & 0 & 1 \end{pmatrix}$
a''	ω^k	$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \omega \end{pmatrix}$	$\begin{pmatrix} \omega & 0 & 0 \\ 0 & \omega & 0 \\ 0 & 0 & \omega^2 \end{pmatrix}$	$\begin{pmatrix} \omega^2 & 0 & 0 \\ 0 & \omega^2 & 0 \\ 0 & 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 & 0 \\ 0 & \omega & 0 \\ 0 & 0 & \omega^2 \end{pmatrix}$
b	ω^ℓ	$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$

Table 46: Representations of a, a', a'' and b in $\Sigma(81)$

13.14 $\Sigma(81)$

All of the $\Sigma(81)$ elements are written as $b^k a^\ell a'^m a''^n$ with $k, \ell, m, n = 0, 1, 2$, where these generators satisfy $a^3 = a'^3 = a''^3 = 1$, $aa' = a'a$, $aa'' = a''a$, $a''a' = a'a''$, $b^3 = 1$, $b^2ab = a''$, $b^2a'b = a$ and $b^2a''b = a'$. Table 46 shows the representations of generators, b, a, a' and a'' on each representation of $\Sigma(81)$.

• $\Sigma(81) \to Z_3 \times Z_3 \times Z_3$

The subgroup $Z_3 \times Z_3 \times Z_3$ consists of $\{e, a, a^2, a', a'^2, a'', a''^2, \cdots\}$. There are 3^3 singlets $\mathbf{1}_{k,\ell,m}$ of $Z_3 \times Z_3 \times Z_3$ and the generators, a, a' and a'', are represented on $\mathbf{1}_{k,\ell,m}$ as $a = \omega^k$,

$a' = \omega^\ell$ and $a'' = \omega^m$. Then, each representation of $\Sigma(81)$ is decomposed as follows,

$$\begin{array}{ccccc} \Sigma(81) & \mathbf{1}_\ell^k & \mathbf{3}_A & \mathbf{3}_B \\ & \downarrow & \downarrow & \downarrow \\ Z_3\times Z_3\times Z_3 & \mathbf{1}_{k,k,k} & \mathbf{1}_{1,0,0}+\mathbf{1}_{0,1,0}+\mathbf{1}_{0,0,1} & \mathbf{1}_{2,1,1}+\mathbf{1}_{1,2,1}+\mathbf{1}_{1,1,2} \end{array},$$

$$\begin{array}{ccc} \Sigma(81) & \mathbf{3}_C & \mathbf{3}_D \\ & \downarrow & \downarrow \\ Z_3\times Z_3\times Z_3 & \mathbf{1}_{0,2,2}+\mathbf{1}_{2,0,2}+\mathbf{1}_{2,2,0} & \mathbf{1}_{2,1,0}+\mathbf{1}_{0,2,1}+\mathbf{1}_{1,0,2} \end{array},$$

$$\begin{array}{ccc} \Sigma(81) & \bar{\mathbf{3}}_A & \bar{\mathbf{3}}_B \\ & \downarrow & \downarrow \\ Z_3\times Z_3\times Z_3 & \mathbf{1}_{2,0,0}+\mathbf{1}_{0,2,0}+\mathbf{1}_{0,0,2} & \mathbf{1}_{1,2,2}+\mathbf{1}_{2,1,2}+\mathbf{1}_{2,2,1} \end{array}, \tag{420}$$

$$\begin{array}{ccc} \Sigma(81) & \bar{\mathbf{3}}_C & \bar{\mathbf{3}}_D \\ & \downarrow & \downarrow \\ Z_3\times Z_3\times Z_3 & \mathbf{1}_{0,1,1}+\mathbf{1}_{1,0,1}+\mathbf{1}_{1,1,0} & \mathbf{1}_{1,2,0}+\mathbf{1}_{0,1,2}+\mathbf{1}_{2,0,1} \end{array}.$$

- $\Sigma(81) \to \Delta(27)$

The subgroup $\Delta(27)$ consists of $b^k\tilde{a}^m\tilde{a}'^n$, where $\tilde{a} = a^2a''$ and $\tilde{a}' = a'a''^2$. Table 47 shows the representations of the generators, b, $\tilde{a}$ and $\tilde{a}'$ on each representation of $\Sigma(81)$. Then, each representation of $\Sigma(81)$ is decomposed to representations of $\Delta(27)$ as

$$\begin{array}{cccccc} \Sigma(81) & \mathbf{1}_\ell^k & \mathbf{3}_A & \mathbf{3}_B & \mathbf{3}_C & \mathbf{3}_D \\ & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\ \Delta(27) & \mathbf{1}_{\ell,0} & \mathbf{3}_{[0][1]} & \mathbf{3}_{[0][1]} & \mathbf{3}_{[0][1]} & \mathbf{1}_{0,2}+\mathbf{1}_{1,2}+\mathbf{1}_{2,2} \end{array},$$

$$\begin{array}{ccccc} \Sigma(81) & \bar{\mathbf{3}}_A & \bar{\mathbf{3}}_B & \bar{\mathbf{3}}_C & \bar{\mathbf{3}}_D \\ & \downarrow & \downarrow & \downarrow & \downarrow \\ \Delta(27) & \mathbf{3}_{[0][2]} & \mathbf{3}_{[0][2]} & \mathbf{3}_{[0][2]} & \mathbf{1}_{0,1}+\mathbf{1}_{1,1}+\mathbf{1}_{2,1} \end{array}. \tag{421}$$

13.15 $\Delta(54)$

All of the $\Delta(54)$ elements are written as $b^kc^\ell a^m a'^n$ with $k, m, n = 0, 1, 2$ and $\ell = 0, 1$. Here, the generators a and a' correspond to Z_3 and Z'_3 of $(Z_3 \times Z'_3) \rtimes S_3$, respectively, while b and c correspond to Z_3 and Z_2 in S_3 of $(Z_3 \times Z'_3) \rtimes S_3$, respectively. Table 48 shows the representations of generators, b, c, a and a' on each representation of $\Delta(54)$.

- $\Delta(54) \to S_3 \times Z_3$

	$\mathbf{1}_\ell^k$	$\mathbf{3}_A$	$\mathbf{3}_B$	$\mathbf{3}_C$	$\mathbf{3}_D$
$\tilde{a}$	1	$\begin{pmatrix} \omega & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \omega^2 \end{pmatrix}$	$\begin{pmatrix} \omega & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \omega^2 \end{pmatrix}$	$\begin{pmatrix} \omega & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \omega^2 \end{pmatrix}$	$\begin{pmatrix} \omega^2 & 0 & 0 \\ 0 & \omega^2 & 0 \\ 0 & 0 & \omega^2 \end{pmatrix}$
$\tilde{a}'$	1	$\begin{pmatrix} \omega^2 & 0 & 0 \\ 0 & \omega & 0 \\ 0 & 0 & 1 \end{pmatrix}$	$\begin{pmatrix} \omega^2 & 0 & 0 \\ 0 & \omega & 0 \\ 0 & 0 & 1 \end{pmatrix}$	$\begin{pmatrix} \omega^2 & 0 & 0 \\ 0 & \omega & 0 \\ 0 & 0 & 1 \end{pmatrix}$	$\begin{pmatrix} \omega^2 & 0 & 0 \\ 0 & \omega^2 & 0 \\ 0 & 0 & \omega^2 \end{pmatrix}$
b	ω^ℓ	$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$

Table 47: Representations of b, $\tilde{a}$ and $\tilde{a}'$ of $\Delta(27)$ in $\Sigma(81)$

	$\mathbf{1}_+$	$\mathbf{1}_-$	$\mathbf{2}_1$	$\mathbf{2}_2$	$\mathbf{2}_3$	$\mathbf{2}_4$	$\mathbf{3}_{1(k)}$	$\mathbf{3}_{2(k)}$
a	1	1	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$	$\begin{pmatrix} \omega^2 & 0 \\ 0 & \omega \end{pmatrix}$	$\begin{pmatrix} \omega & 0 \\ 0 & \omega^2 \end{pmatrix}$	$\begin{pmatrix} \omega & 0 \\ 0 & \omega^2 \end{pmatrix}$	$\begin{pmatrix} \omega^k & 0 & 0 \\ 0 & \omega^{2k} & 0 \\ 0 & 0 & 1 \end{pmatrix}$	$\begin{pmatrix} \omega^k & 0 & 0 \\ 0 & \omega^{2k} & 0 \\ 0 & 0 & 1 \end{pmatrix}$
a'	1	1	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$	$\begin{pmatrix} \omega^2 & 0 \\ 0 & \omega \end{pmatrix}$	$\begin{pmatrix} \omega & 0 \\ 0 & \omega^2 \end{pmatrix}$	$\begin{pmatrix} \omega & 0 \\ 0 & \omega^2 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 & 0 \\ 0 & \omega^k & 0 \\ 0 & 0 & \omega^{2k} \end{pmatrix}$	$\begin{pmatrix} 1 & 0 & 0 \\ 0 & \omega^k & 0 \\ 0 & 0 & \omega^{2k} \end{pmatrix}$
b	1	1	$\begin{pmatrix} \omega & 0 \\ 0 & \omega^2 \end{pmatrix}$	$\begin{pmatrix} \omega & 0 \\ 0 & \omega^2 \end{pmatrix}$	$\begin{pmatrix} \omega & 0 \\ 0 & \omega^2 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$
c	1	-1	$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & -1 \\ 0 & -1 & 0 \\ -1 & 0 & 0 \end{pmatrix}$

Table 48: Representations of a, a', b and c in $\Delta(54)$

The $\Delta(54)$ group includes $S_3 \times Z_3$ as a subgroup. The subgroup S_3 consists of $\{e, b, c, b^2, bc, b^2c\}$. The Z_3 part of $S_3 \times Z_3$ consists of $\{e, aa'^2, a^2a'\}$, where $(aa'^2)^3 = e$ and the element aa'^2 commutes with all of the S_3 elements. Representations, $\boldsymbol{r}_k$, for $S_3 \times Z_3$ are specified by representations $\boldsymbol{r}$ of S_3 and the Z_3 charge k, where $\boldsymbol{r} = \mathbf{1}, \mathbf{1}', \mathbf{2}$ and $k = 0, 1, 2$. That is, the element aa'^2 is represented as $aa'^2 = \omega^k$ on $\boldsymbol{r}_k$ for $k = 0, 1, 2$. For the decomposition of $\Delta(54)$ to $S_3 \times Z_3$, it would be convenient to use the basis for S_3 representations, $\mathbf{1}$, $\mathbf{1}'$ and $\mathbf{2}$, which is shown in Table 49. Then, each representation of $\Delta(54)$ is decomposed to representations of $S_3 \times Z_3$ as follows,

$$\begin{array}{ccccccccc} \Delta(54) & \mathbf{1}_+ & \mathbf{1}_- & \mathbf{2}_1 & \mathbf{2}_2 & \mathbf{2}_3 & \mathbf{2}_4 & \mathbf{3}_{1(k)} & \mathbf{3}_{2(k)} \\ & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\ S_3 & \mathbf{1}_0 & \mathbf{1}'_0 & \mathbf{2}_0 & \mathbf{2}_0 & \mathbf{2}_0 & \mathbf{1}_0 + \mathbf{1}'_0 & \mathbf{1}_k + \mathbf{2}_k & \mathbf{1}'_k + \mathbf{2}_k \end{array}, \tag{422}$$

for $k = 1, 2$. Components of S_3 doublets and singlets obtained from $\Delta(54)$ triplets are

	$\mathbf{1}$	$\mathbf{1}$'	$\mathbf{2}$
b	1	1	$\begin{pmatrix} \omega & 0 \\ 0 & \omega^2 \end{pmatrix}$
c	1	-1	$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

Table 49: Representations of b and c of S_3 in $\Delta(54)$

	$\mathbf{1}_+$	$\mathbf{1}_-$	$\mathbf{2}_1$	$\mathbf{2}_2$	$\mathbf{2}_3$	$\mathbf{2}_4$	$\mathbf{3}_{1(k)}$	$\mathbf{3}_{2(k)}$
$\tilde{a}$	1	1	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$	$\begin{pmatrix} \omega & 0 \\ 0 & \omega^2 \end{pmatrix}$	$\begin{pmatrix} \omega^2 & 0 \\ 0 & \omega \end{pmatrix}$	$\begin{pmatrix} \omega^2 & 0 \\ 0 & \omega \end{pmatrix}$	$\begin{pmatrix} \omega^{2k} & 0 & 0 \\ 0 & \omega^k & 0 \\ 0 & 0 & 1 \end{pmatrix}$	$\begin{pmatrix} \omega^{2k} & 0 & 0 \\ 0 & \omega^k & 0 \\ 0 & 0 & 1 \end{pmatrix}$
a'	1	1	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$	$\begin{pmatrix} \omega^2 & 0 \\ 0 & \omega \end{pmatrix}$	$\begin{pmatrix} \omega & 0 \\ 0 & \omega^2 \end{pmatrix}$	$\begin{pmatrix} \omega & 0 \\ 0 & \omega^2 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 & 0 \\ 0 & \omega^k & 0 \\ 0 & 0 & \omega^{2k} \end{pmatrix}$	$\begin{pmatrix} 1 & 0 & 0 \\ 0 & \omega^k & 0 \\ 0 & 0 & \omega^{2k} \end{pmatrix}$
$\tilde{b}$	1	-1	$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & -1 \\ 0 & -1 & 0 \\ -1 & 0 & 0 \end{pmatrix}$

Table 50: Representations of $\tilde{b}$, $\tilde{a}$ and a' of $\Sigma(18)$ in $\Delta(54)$

the same ones as considered in the decomposition for $S_4 \to S_3$.

- $\Delta(54) \to \Sigma(18)$

We consider the subgroup $\Sigma(18)$, which consists of $\tilde{b}^\ell \tilde{a}^m a'^n$ with $\ell = 0, 1$ and $m, n = 0, 1, 2$, where $\tilde{b} = c$ and $\tilde{a} = a^2$. Table 50 shows the representations of the generators, $\tilde{a}$, a' and $\tilde{b}$ on each representation of $\Delta(54)$. Then, each representation of $\Delta(54)$ is decomposed to representations of $\Sigma(18)$ as follows,

$$\begin{array}{ccccccccc} \Delta(54) & \mathbf{1}_+ & \mathbf{1}_- & \mathbf{2}_1 & \mathbf{2}_2 & \mathbf{2}_3 & \mathbf{2}_4 & \mathbf{3}_{1(k)} & \mathbf{3}_{2(k)} \\ & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\ \Sigma(18) & \mathbf{1}_{+0} & \mathbf{1}_{-0} & \mathbf{2}_{0,0} & \mathbf{2}_{2,1} & \mathbf{2}_{1,2} & \mathbf{2}_{1,2} & \mathbf{1}_{+k} + \mathbf{2}_{0,2k} & \mathbf{1}_{-k} + \mathbf{2}_{2k,0} \end{array} . \tag{423}$$

The decomposition of triplet components is obtained as follows,

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}_{\mathbf{3}_{1(k)}} \to (x_2)_{\mathbf{1}_{+k}} \oplus \begin{pmatrix} x_1 \\ x_3 \end{pmatrix}_{\mathbf{2}_{0,2k}}, \quad \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}_{\mathbf{3}_{2(k)}} \to (x_2)_{\mathbf{1}_{-k}} \oplus \begin{pmatrix} x_3 \\ -x_1 \end{pmatrix}_{\mathbf{2}_{2k,0}}. \tag{424}$$

- $\Delta(54) \to \Delta(27)$

We consider the subgroup $\Delta(27)$, which consists of $b^k a^m a'^n$ with $k, m, n = 0, 1, 2$. By use of Table 48, it is found that each representation of $\Delta(54)$ is decomposed to representations of $\Delta(27)$ as follows,

$$\begin{array}{ccccccccc} \Delta(54) & \mathbf{1}_+ & \mathbf{1}_- & \mathbf{2}_1 & \mathbf{2}_2 & \mathbf{2}_3 & \mathbf{2}_4 & \mathbf{3}_{1(k)} & \mathbf{3}_{2(k)} \\ & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\ \Delta(27) & \mathbf{1}_{0,0} & \mathbf{1}_{0,0} & \mathbf{1}_{1,0}+\mathbf{1}_{2,0} & \mathbf{1}_{1,1}+\mathbf{1}_{2,2} & \mathbf{1}_{1,2}+\mathbf{1}_{2,1} & \mathbf{1}_{0,2}+\mathbf{1}_{0,1} & \mathbf{3}_{[0][k]} & \mathbf{3}_{[0][k]} \end{array} . \quad (425)$$

14 Anomalies

14.1 Generic aspects

In section 15, some phenomenological applications of non-Abelian discrete symmetries are shown as flavor symmetries. In general, symmetries at the tree-level can be broken by quantum effects, i.e. anomalies. Anomalies of continuous symmetries, in particular gauge symmetries, have been studied well. Here we review about anomalies of non-Abelian discrete symmetries. For our purpose, the path integral approach is convenient. Thus, we use Fujikawa's method [213, 214] to derive anomalies of discrete symmetries. (See e.g. Ref. [210].)

Let us consider a gauge theory with a (non-Abelian) gauge group G_g and a set of fermions $\Psi = [\psi^{(1)}, \cdots, \psi^{(M)}]$. Then, we assume that their Lagrangian is invariant under the following chiral transformation,

$$\Psi(x) \to U\Psi(x), \tag{426}$$

with $U = \exp(i\alpha P_L)$ and $\alpha = \alpha^A T_A$, where T_A denote the generators of the transformation and P_L is the left-chiral projector. Here, the above transformation is not necessary a gauge transformation. The fermions $\Psi(x)$ are the (irreducible) M-plet representation $\boldsymbol{R}^M$. For the moment, we suppose that $\Psi(x)$ correspond to a (non-trivial) singlet under the flavor symmetry while they correspond to the $\boldsymbol{R}^M$ representation under the gauge group G_g. Since the generator T_A as well as α is represented on $\boldsymbol{R}^M$ as a $(M \times M)$ matrix, we use the notation, $T_A(\boldsymbol{R}^M)$ and $\alpha(\boldsymbol{R}^M) = \alpha^A T_A(\boldsymbol{R}^M)$.

The anomaly appears in Fujikawa's method from the transformation of the path integral measure as the Jacobian, $J(\alpha)$, i.e.,

$$\mathcal{D}\Psi\mathcal{D}\bar{\Psi} \to \mathcal{D}\Psi\mathcal{D}\bar{\Psi} J(\alpha), \tag{427}$$

where

$$J(\alpha) = \exp\left(i \int d^4x \mathcal{A}(x;\alpha) \right). \tag{428}$$

The anomaly function $\mathcal{A}$ decomposes into a gauge part and a gravitational part [215, 216, 217]

$$\mathcal{A} \;=\; \mathcal{A}_{\text{gauge}} + \mathcal{A}_{\text{grav}} \,. \tag{429}$$

The gauge part is given by

$$\mathcal{A}_{\text{gauge}}(x;\alpha) \;=\; \frac{1}{32\,\pi^2}\text{Tr}\left[\alpha(\boldsymbol{R}^M)\, F^{\mu\nu}(x)\, \widetilde{F}_{\mu\nu}(x)\right] \,, \tag{430}$$

where $F^{\mu\nu}$ denotes the field strength of the gauge fields, $F_{\mu\nu} = [D_\mu, D_\nu]$, and $\widetilde{F}_{\mu\nu}$ denotes its dual, $\widetilde{F}^{\mu\nu} = \varepsilon^{\mu\nu\rho\sigma} F_{\rho\sigma}$. The trace 'Tr' runs over all internal indices. When the transformation corresponds to a continuous symmetry, this anomaly can be calculated by the triangle diagram with external lines of two gauge bosons and one current corresponding to the symmetry for Eq. (426).

Similarly, the gravitation part is obtained as [215, 216, 217]

$$\mathcal{A}_{\rm grav} \;=\; -\,\mathcal{A}_{\rm grav}^{\rm Weyl\;fermion}{\rm tr}\left[\alpha(\boldsymbol{R}^{(M)})\right]\;, \tag{431}$$

where ‘tr’ is the trace for the matrix $(M \times M)$ $T_A(\boldsymbol{R}^M)$. The contribution of a single Weyl fermion to the gravitational anomaly is given by [215, 216, 217]

$$\mathcal{A}_{\rm grav}^{\rm Weyl\;fermion} \;=\; \frac{1}{384\pi^2}\,\frac{1}{2}\,\varepsilon^{\mu\nu\rho\sigma}\,R_{\mu\nu}{}^{\lambda\gamma}\,R_{\rho\sigma\lambda\gamma}\;. \tag{432}$$

When other sets of M_i-plet fermions Ψ_{M_i} are included in a theory, the total gauge and gravity anomalies are obtained as their summations, $\sum_{\Psi_{M_i}} \mathcal{A}_{\rm gauge}$ and $\sum_{\Psi_{M_i}} \mathcal{A}_{\rm grav}$.

For the evaluation of these anomalies, it is useful to recall the index theorems [215, 216], which imply

$$\int d^4x\,\frac{1}{32\pi^2}\,\varepsilon^{\mu\nu\rho\sigma}\,F^a_{\mu\nu}\,F^b_{\rho\sigma}\,{\rm tr}\left[\mathsf{t}_a\,\mathsf{t}_b\right] \;\in\; \mathbb{Z}\;, \tag{433a}$$

$$\frac{1}{2}\int d^4x\,\frac{1}{384\pi^2}\,\frac{1}{2}\,\varepsilon^{\mu\nu\rho\sigma}\,R_{\mu\nu}{}^{\lambda\gamma}\,R_{\rho\sigma\lambda\gamma} \;\in\; \mathbb{Z}\;, \tag{433b}$$

where t_a are generators of G_g in the fundamental representation. We use the convention that ${\rm tr}[\mathsf{t}_a\,\mathsf{t}_b] = \frac{1}{2}\delta_{ab}$. The factor $\frac{1}{2}$ in Eq. (433b) follows from Rohlin’s theorem [218], as discussed in [203]. Of course, these indices are independent of each other. The path integral includes all possible configurations corresponding to different index numbers.

First of all, we study anomalies of the continuous $U(1)$ symmetry. We consider a theory with a (non-Abelian) gauge symmetry G_g as well as the continuous $U(1)$ symmetry, which may be gauged. This theory include fermions with $U(1)$ charges, $q^{(f)}$ and representations $\boldsymbol{R}^{(f)}$. Those anomalies vanish if and only if the Jacobian is trivial, i.e. $J(\alpha) = 1$ for an arbitrary value of α. Using the index theorems, one can find that the anomaly-free conditions require

$$A_{U(1)-G_g-G_g} \;\equiv\; \sum_{\boldsymbol{R}^{(f)}} q^{(f)}\,T_2(\boldsymbol{R}^{(f)}) \;=\; 0, \tag{434}$$

for the mixed $U(1) - G_g - G_g$ anomaly, and

$$A_{U(1)-{\rm grav}-{\rm grav}} \;\equiv\; \sum_f q^{(f)} \;=\; 0, \tag{435}$$

for the $U(1)$–gravity–gravity anomaly. Here, $T_2(\boldsymbol{R}^{(f)})$ is the Dynkin index of the $\boldsymbol{R}^f$ representation, i.e.

$${\rm tr}\left[\mathsf{t}_a\left(\boldsymbol{R}^{(f)}\right)\,\mathsf{t}_b\left(\boldsymbol{R}^{(f)}\right)\right] \;=\; \delta_{ab}T_2(\boldsymbol{R}^{(f)})\;. \tag{436}$$

Next, let us study anomalies of the Abelian discrete symmetry, i.e. the Z_N symmetry. For the Z_N symmetry, we write $\alpha = 2\pi Q_N/N$, where Q_N is the Z_N charge operator and its eigenvalues are integers. Here we denote Z_N charges of fermions as $q_N^{(f)}$. Then we can evaluate the $Z_N-G_g-G_g$ and Z_N-gravity-gravity anomalies as the above $U(1)$ anomalies. However, the important difference is that α takes a discrete value. Then, the anomaly-free conditions, i.e., $J(\alpha)=1$ for a discrete transformation, require

$$A_{Z_N-G_g-G_g} \;=\; \frac{1}{N}\sum_{\boldsymbol{R}^{(f)}} q^{(f)_N}\left(2\,T_2(\boldsymbol{R}^{(f)})\right) \in \mathbb{Z}\;, \tag{437}$$

for the $Z_N - G_g - G_g$ anomaly, and

$$A_{Z_N-\mathrm{grav}-\mathrm{grav}} \;=\; \frac{2}{N}\sum_{f} q_N^{(f)}\,\dim \boldsymbol{R}^{(f)} \in \mathbb{Z}\;, \tag{438}$$

for the Z_N-gravity-gravity anomaly. These anomaly-free conditions reduce to

$$\sum_{\boldsymbol{R}^{(f)}} q_N^{(f)}\,T_2(\boldsymbol{R}^{(f)}) \;=\; 0 \quad \mathrm{mod}\ N/2\;, \tag{439a}$$

$$\sum_{f} q_N^{(f)}\,\dim \boldsymbol{R}^{(f)} \;=\; 0 \quad \mathrm{mod}\ N/2\;. \tag{439b}$$

Note that the Z_2 symmetry is always free from the Z_2-gravity-gravity anomaly.

Finally, we study anomalies of non-Abelian discrete symmetries G [208, 210]. A discrete group G consists of the finite number of elements, g_i. Hence, the non-Abelian discrete symmetry is anomaly-free if and only if the Jacobian is vanishing for the transformation corresponding to each element g_i. Furthermore, recall that $(g_i)^{N_i}=1$. That is, each element g_i in the non-Abelian discrete group generates a Z_{N_i} symmetry. Thus, the analysis on non-Abelian discrete anomalies reduces to one on Abelian discrete anomalies. One can take the field basis such that g_i is represented in a diagonal form. In such a basis, each field has a definite Z_{N_i} charge, $q_{N_i}^{(f)}$. The anomaly-free conditions for the g_i transformation are written as

$$\sum_{\boldsymbol{R}^{(f)}} q_{N_i}^{(f)}\,T_2(\boldsymbol{R}^{(f)}) \;=\; 0 \quad \mathrm{mod}\ N_i/2\;, \tag{440a}$$

$$\sum_{f} q_{N_i}^{(f)}\,\dim \boldsymbol{R}^{(f)} \;=\; 0 \quad \mathrm{mod}\ N_i/2\;. \tag{440b}$$

If these conditions are satisfied for all of $g_i \in G$, there are no anomalies of the full non-Abelian symmetry G. Otherwise, the non-Abelian symmetry is broken completely or partially to its subgroup by quantum effects.

In principle, we can investigate anomalies of non-Abelian discrete symmetries G following the above procedure. However, we give a practically simpler way to analyze those anomalies [208, 210]. Here, we consider again the transformation similar to (426) for a set of fermions $\Psi=[\psi^{(1)},\cdots,\psi^{(Md_\alpha)}]$, which correspond to the $\boldsymbol{R}^M$ irreducible representation of the gauge group G_g and the $\boldsymbol{r}^\alpha$ irreducible representation of the non-Abelian discrete symmetry G with the dimension d_α. Let U correspond to one of group elements $g_i \in G$, which is represented by the matrix $D_\alpha(g_i)$ on $\boldsymbol{r}^\alpha$. Then, the Jacobian is proportional to its determinant, $\det D(g_i)$. Thus, the representations with $\det D_\alpha(g_i) = 1$ do not contribute to anomalies. Therefore, the non-trivial Jacobian, i.e. anomalies are originated from representations with $\det D_\alpha(g_i) \neq 1$. Note that $\det D_\alpha(g_i) = \det D_\alpha(g g_i g^{-1})$ for $g \in G$, that is, the determinant is constant in a conjugacy class. Thus, it would be useful to calculate the determinants of elements on each irreducible representation. Such a determinant for the conjugacy class C_i can be written by

$$\det(C_i)_\alpha = e^{2\pi i q^\alpha_{\hat{N}_i}/\hat{N}_i}, \tag{441}$$

on the irreducible representation $\boldsymbol{r}^\alpha$. Note that $\hat{N}_i$ is a divisor of N_i, where N_i is the order of g_i in the conjugacy class C_i, i.e. $g^{N_i} = e$, such that $q^\alpha_{\hat{N}_i}$ are normalized to be integers for all of the irreducible representations $\boldsymbol{r}^\alpha$. We consider the $Z_{\hat{N}_i}$ symmetries and their anomalies. Then, we obtain the anomaly-free conditions similar to (440). That is, the anomaly-free conditions for the conjugacy classes C_i are written as

$$\sum_{\boldsymbol{r}^{(\alpha)},\boldsymbol{R}^{(f)}} q^{\alpha(f)}_{\hat{N}_i} T_2(\boldsymbol{R}^{(f)}) \ = \ 0 \mod \hat{N}_i/2 \ , \tag{442a}$$

$$\sum_{\alpha.f} q^{\alpha(f)}_{\hat{N}_i} \dim \boldsymbol{R}^{(f)} \ = \ 0 \mod \hat{N}_i/2 \ , \tag{442b}$$

for the theory including fermions with the $\boldsymbol{R}^{(f)}$ representations of the gauge group G_g and the $\boldsymbol{r}^{\alpha(f)}$ representations of the flavor group G, which correspond to the $Z_{\hat{N}_i}$ charges, $q^{\alpha(f)}_{\hat{N}_i}$. Note that the fermion fields with the d_α-dimensional representation $\boldsymbol{r}^\alpha$ contribute to these anomalies, $q^{\alpha(f)}_{\hat{N}_i} T_2(\boldsymbol{R}^{(f)})$ and $q^{\alpha(f)}_{\hat{N}_i} \dim \boldsymbol{R}^{(f)}$, but not $d_\alpha q^{\alpha(f)}_{\hat{N}_i} T_2(\boldsymbol{R}^{(f)})$ and $d_\alpha q^{\alpha(f)}_{\hat{N}_i} \dim \boldsymbol{R}^{(f)}$. If these conditions are satisfied for all of conjugacy classes of G, the full non-Abelian symmetry G is free from anomalies. Otherwise, the non-Abelian symmetry is broken by quantum effects. As we will see below, in concrete examples, the above anomaly-free conditions often lead to the same conditions between different conjugacy classes. Note, when $\hat{N}_i = 2$, the symmetry is always free from the mixed gravitational anomalies. We study explicitly more for concrete groups in what follows.

14.2 Explicit calculations

Here, we apply the above studies on anomalies for concrete groups.

- $\mathbf{S_3}$

We start with S_3. As shown in section 3.1, the S_3 group has the three conjugacy classes, $C_1 = \{e\}$, $C_2 = \{ab, ba\}$ and $C_3 = \{a, b, bab\}$, and three irreducible representations, $\mathbf{1}$, $\mathbf{1}'$ and $\mathbf{2}$. Note that the determinants of elements are constant in a conjugacy class. The determinants of elements in singlet representations are equal to characters. Obviously, the determinants of elements in a trivial singlet representation $\mathbf{1}$ are always equal to 1.

	$\mathbf{1}$	$\mathbf{1}'$	$\mathbf{2}$
$\det(C_1)$	1	1	1
$\det(C_2)$	1	1	1
$\det(C_3)$	1	-1	-1

Table 51: Determinants on S_3 representations

On the doublet representation $\mathbf{2}$, the determinants of representation matrices in C_1, C_2 and C_3 are obtained as 1, 1 and -1, respectively. These determinants are shown in Table 51.

From these results, it is found that only the conjugacy class C_3 is relevant to anomalies and the only Z_2 symmetry can be anomalous. Under such a Z_2 symmetry, the trivial singlet has vanishing Z_2 charge, while the other representations, $\mathbf{1}'$ and $\mathbf{2}$ have the Z_2 charges $q_2 = 1$, that is,

$$\begin{aligned} Z_2 \text{ even} \quad &: \quad \mathbf{1}, \\ Z_2 \text{ odd} \quad &: \quad \mathbf{1}', \quad \mathbf{2}. \end{aligned} \tag{443}$$

Thus, the anomaly-free conditions for the $Z_2 - G_g - G_g$ mixed anomaly (442) are written as

$$\sum_{\mathbf{1}'} \sum_{\boldsymbol{R}^{(f)}} T_2(\boldsymbol{R}^{(f)}) + \sum_{\mathbf{2}} \sum_{\boldsymbol{R}^{(f)}} T_2(\boldsymbol{R}^{(f)}) \;=\; 0 \mod 1 \,. \tag{444}$$

Note that a doublet $\mathbf{2}$ contributes on the anomaly coefficient by not $2T_2(\boldsymbol{R}^{(f)})$ but $T_2(\boldsymbol{R}^{(f)})$, which is the same as $\mathbf{1}'$. To show this explicitly, we have written the summations on $\mathbf{1}'$ and $\mathbf{2}$ separately.

- **$\mathbf{S_4}$**

Similarly, we can study anomalies of S_4. As seen in section 3.2, the S_4 group has five the conjugacy classes, C_1, C_3, C_6, C_6' and C_8 and the five irreducible representations, $\mathbf{1}$, $\mathbf{1}'$, $\mathbf{2}$, $\mathbf{3}$ and $\mathbf{3}'$. The determinants of group elements in each representation are shown in Table 52. These results imply that only the Z_2 symmetry can be anomalous. Under such a Z_2 symmetry, each representation has the following behaviors,

$$\begin{aligned} Z_2 \text{ even} \quad &: \quad \mathbf{1}, \quad \mathbf{3}', \\ Z_2 \text{ odd} \quad &: \quad \mathbf{1}', \quad \mathbf{2}, \quad \mathbf{3}. \end{aligned} \tag{445}$$

Then, the anomaly-free conditions for the $Z_2 - G_g - G_g$ mixed anomaly (440) are written as

$$\sum_{\mathbf{1}'} \sum_{\boldsymbol{R}^{(f)}} T_2(\boldsymbol{R}^{(f)}) + \sum_{\mathbf{2}} \sum_{\boldsymbol{R}^{(f)}} T_2(\boldsymbol{R}^{(f)}) + \sum_{\mathbf{3}} \sum_{\boldsymbol{R}^{(f)}} T_2(\boldsymbol{R}^{(f)}) \;=\; 0 \mod 1 \,. \tag{446}$$

	1	**1'**	**2**	**3**	**3'**
$\det(C_1)$	1	1	1	1	1
$\det(C_3)$	1	1	1	1	1
$\det(C_6)$	1	-1	-1	-1	1
$\det(C_6')$	1	-1	-1	-1	1
$\det(C_8)$	1	1	1	1	1

Table 52: Determinants on S_4 representations

	1	**1'**	**1''**	**3**
$\det(C_1)$	1	1	1	1
$\det(C_3)$	1	1	1	1
$\det(C_4)$	1	ω	ω^2	1
$\det(C_4')$	1	ω^2	ω	1

Table 53: Determinants on A_4 representations

- $\mathbf{A_4}$

We study anomalies of A_4. As shown in section 4.1, there are four conjugacy classes, C_1, C_3, C_4 and C_4', and four irreducible representations, $\mathbf{1}, \mathbf{1}', \mathbf{1}''$ and $\mathbf{3}$. The determinants of group elements in each representation are shown in Table 53, where $\omega = e^{2\pi i/3}$. These results imply that only the Z_3 symmetry can be anomalous. Under such a Z_3 symmetry, each representation has the following Z_3 charge q_3,

$$\begin{aligned} q_3 = 0 &\ :\ \mathbf{1}, \quad \mathbf{3}, \\ q_3 = 1 &\ :\ \mathbf{1}', \\ q_3 = 2 &\ :\ \mathbf{1}''. \end{aligned} \tag{447}$$

This corresponds to the Z_3 symmetry for the conjugacy class C_4. There is another Z_3 symmetry for the conjugacy class C_4', but it is not independent of the former Z_3. Then, the anomaly-free conditions are written as

$$\sum_{\mathbf{1}'} \sum_{\boldsymbol{R}^{(f)}} T_2(\boldsymbol{R}^{(f)}) + 2 \sum_{\mathbf{1}''} \sum_{\boldsymbol{R}^{(f)}} T_2(\boldsymbol{R}^{(f)}) \ = \ 0 \mod 3/2 \,, \tag{448}$$

for the $Z_3 - G_g - G_g$ anomaly and

$$\sum_{\mathbf{1}'} \sum_{\boldsymbol{R}^{(f)}} \dim \boldsymbol{R}^{(f)} + 2 \sum_{\mathbf{1}''} \sum_{\boldsymbol{R}^{(f)}} \dim \boldsymbol{R}^{(f)} \ = \ 0 \mod 3/2 \,, \tag{449}$$

for the Z_3-gravity-gravity anomaly.

- $\mathbf{A_5}$

	1	**3**	$\mathbf{3'}$	**4**	**5**
$\det(C_1)$	1	1	1	1	1
$\det(C_{15})$	1	1	1	1	1
$\det(C_{20})$	1	1	1	1	1
$\det(C_{12})$	1	1	1	1	1
$\det(C'_{12})$	1	1	1	1	1

Table 54: Determinants on A_5 representations

We study anomalies of A_5. As shown in section 4.2, there are five conjugacy classes, $C_1, C_{15}, C_{20}, C_{12}$ and C'_{12}, and five irreducible representations, $\mathbf{1}, \mathbf{3}, \mathbf{3'}, \mathbf{4}$ and $\mathbf{5}$. The determinants of group elements in each representation are shown in Table 53. That is, the determinants of all the A_5 elements are equal to one on any representation. This result can be understood as follows. All of the A_5 elements are written by products of $s = a$ and $t = bab$. The generators, s and t, are written as real matrices on all of representations, $\mathbf{1}, \mathbf{3}, \mathbf{3'}, \mathbf{4}$ and $\mathbf{5}$. Thus, it is found $\det(t) = 1$, because $t^5 = e$. Similarly, since $s^2 = b^3 = e$, the possible values are obtained as $\det(s) = \pm 1$ and $\det(b) = \omega^k$ with $k = 0, 1, 2$. By imposing $\det(bab) = \det(t) = 1$, we find $\det(s) = \det(b) = 1$. Thus, it is found that $\det(g) = 1$ for all of the A_5 elements on any representation. Therefore, the A_5 symmetry is always anomaly-free.

- $\mathbf{T'}$

We study anomalies of T'. As shown in section 5, the T' group has seven conjugacy classes, C_1, C'_1, C_4, C'_4, C''_4, C'''_4 and C_6, and seven irreducible representations, $\mathbf{1}$, $\mathbf{1'}$, $\mathbf{1''}$, $\mathbf{2}$, $\mathbf{2'}$, $\mathbf{2''}$ and $\mathbf{3}$. The determinants of group elements on each representation are shown in Table 55. These results imply that only the Z_3 symmetry can be anomalous. Under such a Z_3 symmetry, each representation has the following Z_3 charge q_3,

$$\begin{aligned} q_3 = 0 \quad &: \quad \mathbf{1}, \quad \mathbf{2}, \quad \mathbf{3}, \\ q_3 = 1 \quad &: \quad \mathbf{1'}, \quad \mathbf{2''}, \\ q_3 = 2 \quad &: \quad \mathbf{1''} \quad \mathbf{2'}. \end{aligned} \tag{450}$$

This corresponds to the Z_3 symmetry for the conjugacy class C_4. There is other Z_3 symmetries for the conjugacy classes C'_4, C''_4 and C'''_4, but those are not independent of the former Z_3. Then, the anomaly-free conditions are written as

$$\begin{aligned} \sum_{\mathbf{1'}} \sum_{\boldsymbol{R}^{(f)}} T_2(\boldsymbol{R}^{(f)}) + 2 \sum_{\mathbf{1''}} \sum_{\boldsymbol{R}^{(f)}} T_2(\boldsymbol{R}^{(f)}) + \sum_{\mathbf{2''}} \sum_{\boldsymbol{R}^{(f)}} T_2(\boldsymbol{R}^{(f)}) \\ + 2 \sum_{\mathbf{2'}} \sum_{\boldsymbol{R}^{(f)}} T_2(\boldsymbol{R}^{(f)}) \ = \ 0 \mod 3/2 \,, \end{aligned} \tag{451}$$

	$\mathbf{1}$	$\mathbf{1}'$	$\mathbf{1}''$	$\mathbf{2}$	$\mathbf{2}'$	$\mathbf{2}''$	$\mathbf{3}$
$\det(C_1)$	1	1	1	1	1	1	1
$\det(C_1')$	1	1	1	1	1	1	1
$\det(C_4)$	1	ω	ω^2	1	ω^2	ω	1
$\det(C_4')$	1	ω^2	ω	1	ω	ω^2	1
$\det(C_4'')$	1	ω	ω^2	1	ω^2	ω	1
$\det(C_4''')$	1	ω^2	ω	1	ω	ω^2	1
$\det(C_6)$	1	1	1	1	1	1	1

Table 55: Determinants on T' representations

for the $Z_3 - G_g - G_g$ anomaly and

$$\sum_{\mathbf{1}'}\sum_{\boldsymbol{R}^{(f)}} \dim \boldsymbol{R}^{(f)} + 2\sum_{\mathbf{1}''}\sum_{\boldsymbol{R}^{(f)}} \dim \boldsymbol{R}^{(f)} + \sum_{\mathbf{2}''}\sum_{\boldsymbol{R}^{(f)}} \dim \boldsymbol{R}^{(f)} + 2\sum_{\mathbf{2}'}\sum_{\boldsymbol{R}^{(f)}} \dim \boldsymbol{R}^{(f)} = 0 \mod 3/2 , \tag{452}$$

for the Z_3-gravity-gravity anomaly.

- $\mathbf{D_N}$ **(N =even)**

We study anomalies of D_N with $N =$ even. As shown in section 6, the D_N group with $N =$ even has the four singlets $\mathbf{1}_{\pm\pm}$ and $(N/2-1)$ doublets $\mathbf{2}_k$. All of the D_N elements can be written as products of two elements, a and b. Their determinants on $\mathbf{2}_k$ are obtained as $\det(a) = 1$ and $\det(b) = -1$. Similarly, we can obtain determinants of a and b on four singlets, $\mathbf{1}_{\pm\pm}$. Indeed, four singlets are classified by values of $\det(b)$ and $\det(ab)$, that is, $\det(b) = 1$ for $\mathbf{1}_{+\pm}$, $\det(b) = -1$ for $\mathbf{1}_{-\pm}$, $\det(ab) = 1$ for $\mathbf{1}_{\pm+}$ and $\det(ab) = -1$ for $\mathbf{1}_{\pm-}$. Thus, the determinants of b and ab are essential for anomalies. Those determinants are summarized in Table 56. This implies that two Z_2 symmetries can be anomalous. One Z_2 corresponds to b and the other Z_2' corresponds to ab. Under these $Z_2 \times Z_2'$ symmetry, each representation has the following behavior,

$$\begin{aligned} Z_2 \text{ even} &: \mathbf{1}_{+\pm}, \\ Z_2 \text{ odd} &: \mathbf{1}_{-\pm}, \quad \mathbf{2}_k, \end{aligned} \tag{453}$$

$$\begin{aligned} Z_2' \text{ even} &: \mathbf{1}_{\pm+}, \\ Z_2' \text{ odd} &: \mathbf{1}_{\pm-}, \quad \mathbf{2}_k. \end{aligned} \tag{454}$$

Then, the anomaly-free conditions are written as

$$\sum_{\mathbf{1}_{-\pm}}\sum_{\boldsymbol{R}^{(f)}} T_2(\boldsymbol{R}^{(f)}) + \sum_{\mathbf{2}_k}\sum_{\boldsymbol{R}^{(f)}} T_2(\boldsymbol{R}^{(f)}) = 0 \mod 1 , \tag{455}$$

	$\mathbf{1}_{++}$	$\mathbf{1}_{+-}$	$\mathbf{1}_{-+}$	$\mathbf{1}_{--}$	$\mathbf{2}_k$
$\det(b)$	1	1	-1	-1	-1
$\det(ab)$	1	-1	1	-1	-1

Table 56: Determinants on D_N representations for $N =$ even

	$\mathbf{1}_{+}$	$\mathbf{1}_{-}$	$\mathbf{2}_k$
$\det(b)$	1	-1	-1
$\det(a)$	1	1	1

Table 57: Determinants on D_N representations for $N =$ odd

for the $Z_2 - G_g - G_g$ anomaly and

$$\sum_{\mathbf{1}_{\pm -}} \sum_{\boldsymbol{R}^{(f)}} T_2(\boldsymbol{R}^{(f)}) + \sum_{\mathbf{2}_k} \sum_{\boldsymbol{R}^{(f)}} T_2(\boldsymbol{R}^{(f)}) \ = \ 0 \mod 1 \ , \tag{456}$$

for the $Z_2' - G_g - G_g$ anomaly.

- **$\mathbf{D_N}$ (N =odd)**

Similarly, we study anomalies of D_N with $N =$ odd. As shown in section 6, the D_N group with $N =$ odd has the two singlets $\mathbf{1}_\pm$ and $(N-1)/2$ doublets $\mathbf{2}_k$. Similarly to D_N with $N =$ even, all elements of D_N with $N =$ odd are written by products of two elements, a and b. The determinants of a are obtained as $\det(a) = 1$ on all of representations, $\mathbf{1}_\pm$ and $\mathbf{2}_k$. The determinants of b are obtained as $\det b = 1$ on $\mathbf{1}_+$ and $\det(b) = -1$ on $\mathbf{1}_-$ and $\mathbf{2}_k$. These are shown in Table 57. Thus, only the Z_2 symmetry corresponding to b can be anomalous. Under such a Z_2 symmetry, each representation has the following behavior,

$$\begin{aligned} Z_2 \text{ even} &: \ \mathbf{1}_+, \\ Z_2 \text{ odd} &: \ \mathbf{1}_-, \quad \mathbf{2}_k. \end{aligned} \tag{457}$$

Then, the anomaly-free condition is written as

$$\sum_{\mathbf{1}_{-}} \sum_{\boldsymbol{R}^{(f)}} T_2(\boldsymbol{R}^{(f)}) + \sum_{\mathbf{2}_k} \sum_{\boldsymbol{R}^{(f)}} T_2(\boldsymbol{R}^{(f)}) \ = \ 0 \mod 1 \ , \tag{458}$$

for the $Z_2 - G_g - G_g$ anomaly.

- **$\mathbf{Q_N}$ (N = 4n)**

We study anomalies of Q_N with $N = 4n$. As shown in section 7, the Q_N group with $N = 4n$ has four singlets $\mathbf{1}_{\pm\pm}$ and $(N/2-1)$ doublets $\mathbf{2}_k$. All elements of Q_N are written by products of a and b. The determinant of a is obtained as $\det(a) = 1$ on all of doublets,

	$\mathbf{1}_{++}$	$\mathbf{1}_{+-}$	$\mathbf{1}_{-+}$	$\mathbf{1}_{--}$	$\mathbf{2}_{k=\text{ odd}}$	$\mathbf{2}_{k=\text{ even}}$
$\det(b)$	1	1	-1	-1	1	-1
$\det(ab)$	1	-1	1	-1	1	-1

Table 58: Determinants on Q_N representations for $N/2 =$ even

$\mathbf{2}_k$. On the other hand, the determinant of b is obtained as $\det(b) = 1$ on the doublets $\mathbf{2}_k$ with $k =$ odd and $\det(b) = -1$ on the doublets $\mathbf{2}_k$ with $k =$ even. Similarly to D_N with $N =$ even, the four singlets $\mathbf{1}_{\pm\pm}$ are classified by values of $\det(b)$ and $\det(ab)$, that is, $\det(b) = 1$ for $\mathbf{1}_{+\pm}$, $\det(b) = -1$ for $\mathbf{1}_{-\pm}$, $\det(b) = 1$ for $\mathbf{1}_{\pm+}$ and $\det(b) = -1$ for $\mathbf{1}_{\pm-}$. Thus, the determinants of b and ab are essential for anomalies. Those determinants are summarized in Table 58. Similarly to D_N with $N =$ even, two Z_2 symmetries can be anomalous. One Z_2 corresponds to b and the other Z_2' corresponds to ab. Under these $Z_2 \times Z_2'$ symmetry, each representation has the following behavior,

$$\begin{aligned} Z_2 \text{ even} &: \mathbf{1}_{+\pm}, \quad \mathbf{2}_{k=\text{ odd}}, \\ Z_2 \text{ odd} &: \mathbf{1}_{-\pm}, \quad \mathbf{2}_{k=\text{ even}}. \end{aligned} \tag{459}$$

$$\begin{aligned} Z_2' \text{ even} &: \mathbf{1}_{\pm+}, \quad \mathbf{2}_{k=\text{ odd}}, \\ Z_2' \text{ odd} &: \mathbf{1}_{\pm-}, \quad \mathbf{2}_{k=\text{ even}}. \end{aligned} \tag{460}$$

Then, the anomaly-free conditions are written as

$$\sum_{\mathbf{1}_{-\pm}} \sum_{\boldsymbol{R}^{(f)}} T_2(\boldsymbol{R}^{(f)}) + \sum_{\mathbf{2}_{k=\text{ even}}} \sum_{\boldsymbol{R}^{(f)}} T_2(\boldsymbol{R}^{(f)}) = 0 \mod 1 , \tag{461}$$

for the $Z_2 - G_g - G_g$ anomaly and

$$\sum_{\mathbf{1}_{\pm-}} \sum_{\boldsymbol{R}^{(f)}} T_2(\boldsymbol{R}^{(f)}) + \sum_{\mathbf{2}_{k=\text{ even}}} \sum_{\boldsymbol{R}^{(f)}} T_2(\boldsymbol{R}^{(f)}) = 0 \mod 1 , \tag{462}$$

for the $Z_2' - G_g - G_g$ anomaly.

- $\mathbf{Q_N}$ $\mathbf{(N = 4n + 2)}$

Similarly, we study anomalies of Q_N with $N = 4n+2$. As shown in section 7, the Q_N group with $N = 4n + 2$ has four singlets $\mathbf{1}_{\pm\pm}$ and $(N/2 - 1)$ doublets $\mathbf{2}_k$. All elements of Q_N are written by products of a and b. The determinants of a are obtained as $\det(a) = 1$ on all of doublets, $\mathbf{2}_k$. On the other hand, the determinants of b are obtained as $\det(b) = 1$ on the doublets $\mathbf{2}_k$ with $k =$ odd and $\det(b) = -1$ on the doublets $\mathbf{2}_k$ with $k =$ even. For all of singlets, it is found that $\chi_\alpha(a) = \chi_\alpha(b^2)$, i.e. $\det(a) = \det(b^2)$. This implies that the determinants of b are more essential for anomalies than a. Indeed, the determinants of b are obtained as $\det(b) = 1$ on $\mathbf{1}_{++}$, $\det(b) = i$ on $\mathbf{1}_{+-}$, $\det(b) = -i$ on $\mathbf{1}_{-+}$ and $\det(b) = -1$

	$\mathbf{1}_{++}$	$\mathbf{1}_{+-}$	$\mathbf{1}_{-+}$	$\mathbf{1}_{--}$	$\mathbf{2}_{k=\text{ odd}}$	$\mathbf{2}_{k=\text{ even}}$
$\det(b)$	1	i	$-i$	-1	1	-1
$\det(a)$	1	-1	-1	1	1	1

Table 59: Determinants on Q_N representations for $N/2 = $ odd

on $\mathbf{1}_{--}$. Those determinants are summarized in Table 59. This result implies that only the Z_4 symmetry corresponding to b can be anomalous. Under such a Z_4 symmetry, each representation has the following Z_4 charge q_4,

$$\begin{aligned} q_4 = 0 &\ :\ \mathbf{1}_{++}, \quad \mathbf{2}_{k=\text{ odd}}, \\ q_4 = 1 &\ :\ \mathbf{1}_{+-}, \\ q_4 = 2 &\ :\ \mathbf{1}_{--}, \quad \mathbf{2}_{k=\text{ even}}, \\ q_4 = 3 &\ :\ \mathbf{1}_{-+}. \end{aligned} \tag{463}$$

That includes the Z_2 symmetry corresponding to a and the Z_2 charge q_2 for each representation is defined as $q_2 = q_4 \bmod 2$. The anomaly-free conditions are written as

$$\begin{aligned} &\sum_{\mathbf{1}_{+-}} \sum_{\boldsymbol{R}^{(f)}} T_2(\boldsymbol{R}^{(f)}) + 2 \sum_{\mathbf{1}_{--}} \sum_{\boldsymbol{R}^{(f)}} T_2(\boldsymbol{R}^{(f)}) + 3 \sum_{\mathbf{1}_{-+}} \sum_{\boldsymbol{R}^{(f)}} T_2(\boldsymbol{R}^{(f)}) \\ &\qquad +2 \sum_{\mathbf{2}_{k=\text{ even}}} \sum_{\boldsymbol{R}^{(f)}} T_2(\boldsymbol{R}^{(f)}) \ = \ 0 \quad \text{mod } 2 \ , \end{aligned} \tag{464}$$

for the $Z_4 - G_g - G_g$ anomaly and

$$\begin{aligned} &\sum_{\mathbf{1}_{+-}} \sum_{\boldsymbol{R}^{(f)}} \dim \boldsymbol{R}^{(f)} + 2 \sum_{\mathbf{1}_{--}} \sum_{\boldsymbol{R}^{(f)}} \dim \boldsymbol{R}^{(f)} + 3 \sum_{\mathbf{1}_{-+}} \sum_{\boldsymbol{R}^{(f)}} \dim \boldsymbol{R}^{(f)} \\ &\qquad +2 \sum_{\mathbf{2}_{k=\text{ even}}} \sum_{\boldsymbol{R}^{(f)}} \dim \boldsymbol{R}^{(f)} \ = \ 0 \quad \text{mod } 2 \ , \end{aligned} \tag{465}$$

for the Z_4-gravity-gravity anomaly. Similarly, we can obtain the anomaly-free condition on the Z_2 symmetry corresponding to a as

$$\sum_{\mathbf{1}_{+-}} \sum_{\boldsymbol{R}^{(f)}} T_2(\boldsymbol{R}^{(f)}) + \sum_{\mathbf{1}_{-+}} \sum_{\boldsymbol{R}^{(f)}} T_2(\boldsymbol{R}^{(f)}) \ = \ 0 \quad \text{mod } 1 \ , \tag{466}$$

for the $Z_2 - G_g - G_g$ anomaly.

- $\mathbf{\Sigma(2N^2)}$

We study anomalies of $\Sigma(2N^2)$. As shown in section 8, the $\Sigma(2N^2)$ group has $2N$ singlets, $\mathbf{1}_{\pm n}$, and $N(N-1)/2$ doublets, $\mathbf{2}_{p,q}$. All elements of $\Sigma(2N^2)$ can be written by products of a, a' and b. Their determinants for each representation are shown in Table 60, where $\rho = e^{2\pi i/N}$. Then, it is found that only the Z_2 symmetry corresponding to b and the

	$\mathbf{1}_{+n}$	$\mathbf{1}_{-n}$	$\mathbf{2}_k$
$\det(b)$	1	-1	-1
$\det(a)$	ρ^n	ρ^n	ρ^{p+q}
$\det(a')$	ρ^n	ρ^n	ρ^{p+q}

Table 60: Determinants on $\Sigma(2N^2)$ representations

Z_N symmetry corresponding to a can be anomalous. Another Z_N symmetry corresponding to a' is not independent of the Z_N symmetry for a. Under such Z_2 symmetry, each representation has the following behavior,

$$\begin{aligned} Z_2 \text{ even} &: \mathbf{1}_{+n}, \\ Z_2 \text{ odd} &: \mathbf{1}_{-n}, \quad \mathbf{2}_{p,q}, \end{aligned} \tag{467}$$

and under the Z_N symmetry corresponding to a each representation has the following Z_N charge q_N,

$$\begin{aligned} q_N = n &: \mathbf{1}_{\pm n}, \\ q_N = p+q &: \mathbf{2}_{p,q}. \end{aligned} \tag{468}$$

Then, the anomaly-free condition is obtained as

$$\sum_{\mathbf{1}_{-n}} \sum_{\boldsymbol{R}^{(f)}} T_2(\boldsymbol{R}^{(f)}) + \sum_{\mathbf{2}_{p,q}} \sum_{\boldsymbol{R}^{(f)}} T_2(\boldsymbol{R}^{(f)}) = 0 \mod 1 , \tag{469}$$

for the $Z_2 - G_g - G_g$ anomaly. Similarly, the anomaly-free conditions for the Z_N symmetry are obtained as

$$\sum_{\mathbf{1}_{\pm n}} \sum_{\boldsymbol{R}^{(f)}} n T_2(\boldsymbol{R}^{(f)}) + \sum_{\mathbf{2}_{p,q}} \sum_{\boldsymbol{R}^{(f)}} (p+q) T_2(\boldsymbol{R}^{(f)}) = 0 \mod N/2 , \tag{470}$$

for the $Z_N - G_g - G_g$ anomaly and

$$\sum_{\mathbf{1}_{\pm n}} \sum_{\boldsymbol{R}^{(f)}} n \dim \boldsymbol{R}^{(f)} + \sum_{\mathbf{2}_{p,q}} \sum_{\boldsymbol{R}^{(f)}} (p+q) \dim \boldsymbol{R}^{(f)} = 0 \mod N/2 , \tag{471}$$

for the Z_N-gravity-gravity anomaly.

- **$\boldsymbol{\Delta(3N^2)}$ (N/3 $\neq$integer)**

We study anomalies of $\Delta(3N^2)$ with $N/3 \neq$ integer. As shown in section 9, the $\Delta(3N^2)$ group with $N/3 \neq$ integer has three singlets, $\mathbf{1}_0$, $\mathbf{1}_1$ and $\mathbf{1}_2$, and $(N^2-1)/3$ triplets, $\mathbf{3}_{[k][\ell]}$. All elements of $\Delta(3N^2)$ can be written by products of a, a' and b. It is found that $\det(a) = \det(a') = 1$ on all of representations. Thus, these elements are irrelevant to anomalies. On the other hand, the determinant of b is obtained as $\det(b) = 1$

	$\mathbf{1}_k$	$\mathbf{3}_{[k][\ell]}$
$\det(b)$	ω^k	1
$\det(a)$	1	1
$\det(a')$	1	1

Table 61: Determinants on $\Delta(3N^2)$ representations ($N/3 \neq$ integer)

for all $\mathbf{3}_{[k][\ell]}$ and $\mathbf{1}_0$, $\det(b) = \omega$ for $\mathbf{1}_1$ and $\det(b) = \omega^2$ for $\mathbf{1}_2$, with $\omega = e^{2\pi i/3}$, as shown in Table 61. This implies that only the Z_3 symmetry corresponding to b can be anomalous. Under such a Z_3 symmetry, each representation has the following Z_3 charge q_3,

$$\begin{aligned} q_3 = 0 &\ :\ \mathbf{1}_0, \quad \mathbf{3}_{[k][\ell]}, \\ q_3 = 1 &\ :\ \mathbf{1}_1, \\ q_3 = 2 &\ :\ \mathbf{1}_2. \end{aligned} \tag{472}$$

Then, the anomaly-free conditions are written as

$$\sum_{\mathbf{1}_1} \sum_{\boldsymbol{R}^{(f)}} T_2(\boldsymbol{R}^{(f)}) + 2 \sum_{\mathbf{1}_2} \sum_{\boldsymbol{R}^{(f)}} T_2(\boldsymbol{R}^{(f)}) \ = \ 0 \quad \mathrm{mod}\ 3/2\ , \tag{473}$$

for the $Z_3 - G_g - G_g$ anomaly and

$$\sum_{\mathbf{1}_1} \sum_{\boldsymbol{R}^{(f)}} \dim \boldsymbol{R}^{(f)} + 2 \sum_{\mathbf{1}_2} \sum_{\boldsymbol{R}^{(f)}} \dim \boldsymbol{R}^{(f)} \ = \ 0 \quad \mathrm{mod}\ 3/2\ , \tag{474}$$

for the Z_3-gravity-gravity anomaly.

- **$\mathbf{\Delta(3N^2)}$ (N/3 =integer)**

Similarly, we can study anomalies of $\Delta(3N^2)$ with $N/3 =$ integer. As shown in section 9, the $\Delta(3N^2)$ group with $N/3 =$ integer has nine singlets $\mathbf{1}_{k\ell}$ and $(N^2-3)/3$ triplets, $\mathbf{3}_{[k][\ell]}$. All elements $\Delta(3N^2)$ can be written by products of a, a' and b. On all of triplet representations $\mathbf{3}_{[k][\ell]}$, their determinants are obtained as $\det(a) = \det(a') = \det(b) = 1$. On the other hand, it is found that $\det(a) = \det(a')$ on all of nine singlets. Furthermore, nine singlets are classified by values of $\det(a) = \det(a')$ and $\det(b)$. That is, the determinants of $\det(a) = \det(a')$ and $\det(b)$ are obtained as $\det(a) = \det(a') = \omega^\ell$ and $\det(b) = \omega^k$ on $\mathbf{1}_{k\ell}$. These results are shown in Table 62. This implies that two independent Z_3 symmetries can be anomalous. One corresponds to b and the other corresponds to a. For the Z_3 symmetry corresponding to b, each representation has the following Z_3 charge q_3,

$$\begin{aligned} q_3 = 0 &\ :\ \mathbf{1}_{0\ell}, \quad \mathbf{3}_{[k][\ell]}, \\ q_3 = 1 &\ :\ \mathbf{1}_{1\ell}, \\ q_3 = 2 &\ :\ \mathbf{1}_{2\ell}, \end{aligned} \tag{475}$$

	$\mathbf{1}_{k\ell}$	$\mathbf{3}_{[k][\ell]}$
$\det(b)$	ω^k	1
$\det(a)$	ω^ℓ	1
$\det(a')$	ω^ℓ	1

Table 62: Determinants on $\Delta(3N^2)$ representations ($N/3 =$ integer)

while for Z_3' symmetry corresponding to a, each representation has the following Z_3 charge q_3',

$$\begin{aligned} q_3' = 0 \quad &: \quad \mathbf{1}_{k0}, \quad \mathbf{3}_{[k][\ell]}, \\ q_3' = 1 \quad &: \quad \mathbf{1}_{k1}, \\ q_3' = 2 \quad &: \quad \mathbf{1}_{k2}. \end{aligned} \tag{476}$$

Then, the anomaly-free conditions are written as

$$\sum_{\mathbf{1}_{1\ell}} \sum_{\boldsymbol{R}^{(f)}} T_2(\boldsymbol{R}^{(f)}) + 2 \sum_{\mathbf{1}_{2\ell}} \sum_{\boldsymbol{R}^{(f)}} T_2(\boldsymbol{R}^{(f)}) \;=\; 0 \quad \mathrm{mod}\ 3/2\ , \tag{477}$$

for the $Z_3 - G_g - G_g$ anomaly and

$$\sum_{\mathbf{1}_{1\ell}} \sum_{\boldsymbol{R}^{(f)}} \dim \boldsymbol{R}^{(f)} + 2 \sum_{\mathbf{1}_{2\ell}} \sum_{\boldsymbol{R}^{(f)}} \dim \boldsymbol{R}^{(f)} \;=\; 0 \quad \mathrm{mod}\ 3/2\ , \tag{478}$$

for the Z_3-gravity-gravity anomaly. Similarly, for the Z_3' symmetry, the anomaly-free conditions are written as

$$\sum_{\mathbf{1}_{k1}} \sum_{\boldsymbol{R}^{(f)}} T_2(\boldsymbol{R}^{(f)}) + 2 \sum_{\mathbf{1}_{k2}} \sum_{\boldsymbol{R}^{(f)}} T_2(\boldsymbol{R}^{(f)}) \;=\; 0 \quad \mathrm{mod}\ 3/2\ , \tag{479}$$

for the $Z_3' - G_g - G_g$ anomaly and

$$\sum_{\mathbf{1}_{k1}} \sum_{\boldsymbol{R}^{(f)}} \dim \boldsymbol{R}^{(f)} + 2 \sum_{\mathbf{1}_{k2}} \sum_{\boldsymbol{R}^{(f)}} \dim \boldsymbol{R}^{(f)} \;=\; 0 \quad \mathrm{mod}\ 3/2\ , \tag{480}$$

for the Z_3'-gravity-gravity anomaly.

- $\mathbf{T_7}$

We study anomalies of T_7. As shown in section 10, the T_7 group has three singlets, $\mathbf{1}_{0,1,2}$, and two triplets, $\mathbf{3}$ and $\bar{\mathbf{3}}$. All elements of T_7 can be written by products of a and b, where a and b correspond to the generators of Z_7 and Z_3, respectively. It is found that $\det(a) = 1$ on all of representations. Thus, these elements are irrelevant to anomalies. On the other hand, the determinant of b is obtained as $\det(b) = 1$ for both $\mathbf{3}$ and $\bar{\mathbf{3}}$ and $\det(b) = \omega^k$ for $\mathbf{1}_k$ ($k = 0, 1, 2$), as shown in Table 63. These results imply that only the

	$\mathbf{1}_0$	$\mathbf{1}_1$	$\mathbf{1}_2$	$\mathbf{3}$	$\bar{\mathbf{3}}$
$\det(a)$	1	1	1	1	1
$\det(b)$	1	ω	ω^2	1	1

Table 63: Determinants on T_7 representations

Z_3 symmetry corresponding to b can be anomalous. Under such a Z_3 symmetry, each representation has the following Z_3 charge q_3,

$$\begin{aligned} q_3 = 0 &\ :\ \mathbf{1}_0, \quad \mathbf{3}, \quad \bar{\mathbf{3}}, \\ q_3 = 1 &\ :\ \mathbf{1}_1, \\ q_3 = 2 &\ :\ \mathbf{1}_2. \end{aligned} \tag{481}$$

Then, the anomaly-free conditions are written as

$$\sum_{\mathbf{1}_1} \sum_{\boldsymbol{R}^{(f)}} T_2(\boldsymbol{R}^{(f)}) + 2 \sum_{\mathbf{1}_2} \sum_{\boldsymbol{R}^{(f)}} T_2(\boldsymbol{R}^{(f)}) \ = \ 0 \quad \mathrm{mod}\ 3/2\ , \tag{482}$$

for the $Z_3 - G_g - G_g$ anomaly and

$$\sum_{\mathbf{1}_1} \sum_{\boldsymbol{R}^{(f)}} \dim \boldsymbol{R}^{(f)} + 2 \sum_{\mathbf{1}_2} \sum_{\boldsymbol{R}^{(f)}} \dim \boldsymbol{R}^{(f)} \ = \ 0 \quad \mathrm{mod}\ 3/2\ , \tag{483}$$

for the Z_3-gravity-gravity anomaly.

- **$\boldsymbol{\Sigma}$(81)**

We study anomalies of $\Sigma(81)$. As shown in section 11, the $\Sigma(81)$ group has nine singlets, $\mathbf{1}_\ell^k$, and eight triplets, $\mathbf{3}_{A,B,C,D}$ and $\bar{\mathbf{3}}_{A,B,C,D}$. All elements of $\Sigma(81)$ can be written by products of a, a', a'', and b. The determinants of those elements on each representation are shown in Table 64. These results imply that there are two independent Z_3 symmetries, which can be anomalous. One is the Z_3 symmetry corresponding to b and the other is the Z_3 symmetry corresponding to a. For the Z_3 symmetry corresponding to b, each representation has the following Z_3 charge q_3,

$$\begin{aligned} q_3 = 0 &\ :\ \mathbf{1}_0^k, \mathbf{3}_{A,B,C,D}, \bar{\mathbf{3}}_{A,B,C,D}, \\ q_3 = 1 &\ :\ \mathbf{1}_1^k, \\ q_3 = 2 &\ :\ \mathbf{1}_2^k, \end{aligned} \tag{484}$$

while for the Z_3' symmetry corresponding to a each representation has the following Z_3' charge q_3',

$$\begin{aligned} q_3' = 0 &\ :\ \mathbf{1}_\ell^0, \mathbf{3}_D, \bar{\mathbf{3}}_D, \\ q_3' = 1 &\ :\ \mathbf{1}_\ell^1, \mathbf{3}_{A,B,C}, \\ q_3' = 2 &\ :\ \mathbf{1}_\ell^2, \bar{\mathbf{3}}_{A,B,C}. \end{aligned} \tag{485}$$

	$\mathbf{1}_\ell^k$	$\mathbf{3}_A$	$\bar{\mathbf{3}}_A$	$\mathbf{3}_B$	$\bar{\mathbf{3}}_B$	$\mathbf{3}_C$	$\bar{\mathbf{3}}_C$	$\mathbf{3}_D$	$\bar{\mathbf{3}}_D$
$\det(b)$	ω^ℓ	1	1	1	1	1	1	1	1
$\det(a)$	ω^k	ω	ω^2	ω	ω^2	ω	ω^2	1	1
$\det(a')$	ω^k	ω	ω^2	ω	ω^2	ω	ω^2	1	1
$\det(a'')$	ω^k	ω	ω^2	ω	ω^2	ω	ω^2	1	1

Table 64: Determinants on $\Sigma(81)$ representations

Then, the anomaly-free conditions are written as

$$\sum_{\mathbf{1}_1^k}\sum_{\boldsymbol{R}^{(f)}} T_2(\boldsymbol{R}^{(f)}) + 2\sum_{\mathbf{1}_2^k}\sum_{\boldsymbol{R}^{(f)}} T_2(\boldsymbol{R}^{(f)}) \;=\; 0 \mod 3/2\,, \tag{486}$$

for the $Z_3 - G_g - G_g$ anomaly and

$$\sum_{\mathbf{1}_1^k}\sum_{\boldsymbol{R}^{(f)}} \dim \boldsymbol{R}^{(f)} + 2\sum_{\mathbf{1}_2^k}\sum_{\boldsymbol{R}^{(f)}} \dim \boldsymbol{R}^{(f)} \;=\; 0 \mod 3/2\,, \tag{487}$$

for the Z_3-gravity-gravity anomaly. Similarly, for the Z_3' symmetry corresponding to a, the anomaly-free conditions are written as

$$\begin{aligned}\sum_{\mathbf{1}_\ell^1}\sum_{\boldsymbol{R}^{(f)}} T_2(\boldsymbol{R}^{(f)}) + \sum_{\mathbf{3}_{A,B,C}}\sum_{\boldsymbol{R}^{(f)}} T_2(\boldsymbol{R}^{(f)}) + 2\sum_{\mathbf{1}_\ell^2}\sum_{\boldsymbol{R}^{(f)}} T_2(\boldsymbol{R}^{(f)}) \\ + 2\sum_{\bar{\mathbf{3}}_{A,B,C}}\sum_{\boldsymbol{R}^{(f)}} T_2(\boldsymbol{R}^{(f)}) \;=\; 0 \mod 3/2\,,\end{aligned} \tag{488}$$

for the $Z_3' - G_g - G_g$ anomaly and

$$\begin{aligned}\sum_{\mathbf{1}_\ell^1}\sum_{\boldsymbol{R}^{(f)}} \dim \boldsymbol{R}^{(f)} + \sum_{\mathbf{3}_{A,B,C}}\sum_{\boldsymbol{R}^{(f)}} \dim \boldsymbol{R}^{(f)} + 2\sum_{\mathbf{1}_\ell^2}\sum_{\boldsymbol{R}^{(f)}} \dim \boldsymbol{R}^{(f)} \\ + 2\sum_{\bar{\mathbf{3}}_{A,B,C}}\sum_{\boldsymbol{R}^{(f)}} \dim \boldsymbol{R}^{(f)} \;=\; 0 \mod 3/2\,,\end{aligned} \tag{489}$$

for the Z_3'-gravity-gravity anomaly.

- **$\Delta(54)$**

We study anomalies of $\Delta(54)$. As shown in section 12, the $\Delta(54)$ group has two singlets, $\mathbf{1}_{+,-}$, four doublets, $\mathbf{2}_{1,2,3,4}$, and four triplets, $\mathbf{3}_{1(k)}$ and $\mathbf{3}_{2(k)}$. All elements of $\Delta(54)$ can be written by products of a, a', b and c. Determinants of a, a' and b on any representation are obtained as $\det(a) = \det(a') = \det(b) = 1$. The determinants of c for $\mathbf{1}_+$ and $\mathbf{3}_2(k)$ are obtained as $\det(c) = 1$ while the other representations lead to $\det(c) = -1$. These results are shown in Table 65. That implies that only the Z_2

	$\mathbf{1}_+$	$\mathbf{1}_-$	$\mathbf{2}_1$	$\mathbf{2}_2$	$\mathbf{2}_3$	$\mathbf{2}_4$	$\mathbf{3}_{1(k)}$	$\mathbf{3}_{2(k)}$
$\det(a)$	1	1	1	1	1	1	1	1
$\det(a')$	1	1	1	1	1	1	1	1
$\det(b)$	1	1	1	1	1	1	1	1
$\det(c)$	1	-1	-1	-1	-1	-1	-1	1

Table 65: Determinants on $\Delta(54)$ representations

symmetry corresponding to the generator c can be anomalous. Under such a Z_2 symmetry, each representation has the following Z_2 charge q_2,

$$\begin{aligned} q_2 = 0 &\quad : \quad \mathbf{1}_+, \mathbf{3}_{2(k)}, \\ q_2 = 1 &\quad : \quad \mathbf{1}_-, \mathbf{2}_{1,2,3,4}, \mathbf{3}_{1(k)}. \end{aligned} \tag{490}$$

Then, the anomaly-free conditions are written as

$$\sum_{\mathbf{1}_-} \sum_{\boldsymbol{R}^{(f)}} T_2(\boldsymbol{R}^{(f)}) + \sum_{\mathbf{2}_k} \sum_{\boldsymbol{R}^{(f)}} T_2(\boldsymbol{R}^{(f)}) + \sum_{\mathbf{3}_{1(k)}} \sum_{\boldsymbol{R}^{(f)}} T_2(\boldsymbol{R}^{(f)}) \ = \ 0 \mod 1 , \tag{491}$$

for the $Z_2 - G_g - G_g$ anomaly.

Similarly, we can analyze on anomalies for other non-Abelian discrete symmetries.

14.3 Comments on anomalies

Finally, we comment on the symmetry breaking effects by quantum effect. When a discrete (flavor) symmetry is anomalous, breaking terms can appear in Lagrangian, e.g. by instanton effects, such as $\frac{1}{M^n}\Lambda^m \Phi_1 \cdots \Phi_k$, where Λ is a dynamical scale and M is a typical (cut-off) scale. Within the framework of string theory discrete anomalies as well as anomalies of continuous gauge symmetries can be canceled by the Green-Schwarz (GS) mechanism [219] unless discrete symmetries are accidental. In the GS mechanism, dilaton and moduli fields, i.e. the so-called GS fields Φ_{GS}, transform non-linearly under anomalous transformation. The anomaly cancellation due to the GS mechanism imposes certain relations among anomalies. (See e.g. Ref. [210].)[2] Stringy non-perturbative effects as well as field-theoretical effects induce terms in Lagrangian such as $\frac{1}{M^n} e^{-a\Phi_{GS}} \Phi_1 \cdots \Phi_k$. The GS fields Φ_{GS}, i.e. dilaton/moduli fields are expected to develop non-vanishing vacuum expectation values and the above terms correspond to breaking terms of discrete symmetries.

The above breaking terms may be small. Such approximate discrete symmetries with small breaking terms may be useful in particle physics,[3] if breaking terms are controllable. Alternatively, if exact symmetries are necessary, one has to arrange matter fields and their quantum numbers such that models are free from anomalies.

[2] See also Ref. [220].

[3] See for some applications e.g. [221].

15 Flavor Models with Non-Abelian Discrete Symmetry

We have shown several group-theoretical aspects for various non-Abelian discrete groups. In this section, we study some phenomenological applications of these discrete symmetries.

15.1 Tri-bimaximal mixing of lepton flavor

The non-Abelian discrete group has been applied to the flavor symmetry of quarks and leptons. Especially, the recent experimental data of neutrinos have encouraged us to work in the non-Abelian discrete symmetry of flavors. The global fit of the neutrino experimental data in Table 66 [3, 4, 5], strongly indicates the tri-bimaximal mixing matrix $U_{\rm tribi}$ for three lepton flavors [7, 8, 9, 10] as follows:

$$U_{\rm tribi} = \begin{pmatrix} \frac{2}{\sqrt{6}} & \frac{1}{\sqrt{3}} & 0 \\ -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} \end{pmatrix}, \tag{492}$$

which favors the non-Abelian discrete symmetry for the lepton flavor. Indeed, various types of models leading to the tri-bimaximal mixing have been proposed by assuming several types of non-Abelian flavor symmetries as seen , e.g. in the review by Altarelli and Feruglio [11]. In this section, we introduce typical models to reproduce the lepton flavor mixing.

parameter	best fit	2σ	3σ	tri-bimaximal
Δm^2_{21} $[10^{-5}{\rm eV}^2]$	$7.59^{+0.23}_{-0.18}$	7.22–8.03	7.03–8.27	*
$\lvert\Delta m^2_{31}\rvert$ $[10^{-3}{\rm eV}^2]$	$2.40^{+0.12}_{-0.11}$	2.18–2.64	2.07–2.75	*
$\sin^2\theta_{12}$	$0.318^{+0.019}_{-0.016}$	0.29–0.36	0.27–0.38	1/3
$\sin^2\theta_{23}$	$0.50^{+0.07}_{-0.06}$	0.39–0.63	0.36–0.67	1/2
$\sin^2\theta_{13}$	$0.013^{+0.013}_{-0.009}$	≤ 0.039	≤ 0.053	0

Table 66: Best-fit values with 1σ errors, and 2σ and 3σ intervals (1 d.o.f) for the three–flavor neutrino oscillation parameters from global data including solar, atmospheric, reactor (KamLAND and CHOOZ) and accelerator (K2K and MINOS) experiments in Ref. [3].

The neutrino mass matrix with the tri-bimaximal mixing of flavors is expressed by the sum of simple mass matrices in the flavor diagonal basis of the charged lepton. In terms of neutrino mass eigenvalues m_1, m_2 and m_3, the neutrino mass matrix is given as

$$M_\nu = U_{\rm tribi}^* \begin{pmatrix} m_1 & 0 & 0 \\ 0 & m_2 & 0 \\ 0 & 0 & m_3 \end{pmatrix} U_{\rm tribi}^\dagger$$
$$= \frac{m_1+m_3}{2}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} + \frac{m_2-m_1}{3}\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} + \frac{m_1-m_3}{2}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}. \quad (493)$$

This neutrino mass matrix can be easily realized in some non-Abelian discrete symmetry. In the following subsection, we present simple realization of this neutrino mass matrix, which arise from the dimension-five non-renormalizable operators [222], or the see-saw mechanism [223, 224, 225, 226, 227].

15.2 A_4 flavor model

Natural models realizing the tri-bimaximal mixing have been proposed based on the non-Abelian finite group A_4 [89]-[151]. The A_4 flavor model considered by Alterelli et al [96, 97] realizes the tri-bimaximal flavor mixing. The deviation from the tri-bimaximal mixing can also be predicted. Actually, one of authors has investigated the deviation from the tri-bimaximal mixing including higher dimensional operators in the effective model [111, 128].

In this subsection, we present the A_4 flavor model with the supersymmetry including the right-handed neutrinos. In the non-Abelian finite group A_4, there are twelve group elements and four irreducible representations: $\mathbf{1}$, $\mathbf{1}'$, $\mathbf{1}''$ and $\mathbf{3}$. The A_4 and Z_3 charge assignments of leptons, Higgs fields and SM-singlets are listed in Table 67. Under the A_4 symmetry, the chiral superfields for three families of the left-handed lepton doublets $l = (l_e, l_\mu, l_\tau)$ and right handed neutrinos $\nu^c = (\nu_e^c, \nu_\mu^c, \nu_\tau^c)$ are assumed to transform as $\mathbf{3}$, while the right-handed ones of the charged lepton singlets e^c, μ^c and τ^c are assigned with $\mathbf{1}$, $\mathbf{1}'$, $\mathbf{1}''$, respectively. The third row of Table 67 shows how each chiral multiplet transforms under Z_3, where $\omega = e^{2\pi i/3}$. The flavor symmetry is spontaneously broken by vacuum expectation values (VEVs) of two $\mathbf{3}$'s, ϕ_T, ϕ_S, and by one singlet, ξ, which are $SU(2)_L \times U(1)_Y$ singlets. Their Z_3 charges are also shown in Table 67. Hereafter, we follow the convention that the chiral superfield and its lowest component are denoted by the same letter.

	(l_e, l_μ, l_τ)	$(\nu_e^c, \nu_\mu^c, \nu_\tau^c)$	e^c	μ^c	τ^c	h_u	h_d	ξ	$(\phi_{T_1}, \phi_{T_2}, \phi_{T_3})$	$(\phi_{S_1}, \phi_{S_2}, \phi_{S_3})$	Φ
A_4	$\mathbf{3}$	$\mathbf{3}$	$\mathbf{1}$	$\mathbf{1}''$	$\mathbf{1}'$	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{3}$	$\mathbf{3}$	$\mathbf{1}$
Z_3	ω	ω^2	ω^2	ω^2	ω^2	1	1	ω^2	1	ω^2	1
$U(1)_{FN}$	0	0	$2q$	q	0	0	0	0	0	0	-1

Table 67: A_4, Z_3 and $U(1)_{FN}$ charges

Allowed terms in the superpotential including charged leptons are written by

$$
\begin{aligned}
w_l &= y_0^e e^c l \phi_T h_d \frac{\Phi^{2q}}{\Lambda'^{2q}} \frac{1}{\Lambda} + y_0^\mu \mu^c l \phi_T h_d \frac{\Phi^q}{\Lambda'^q} \frac{1}{\Lambda} + y_0^\tau \tau^c l \phi_T h_d \frac{1}{\Lambda} \\
&+ y_1^e e^c l \phi_T \phi_T h_d \frac{\Phi^{2q}}{\Lambda'^{2q}} \frac{1}{\Lambda} + y_1^\mu \mu^c l \phi_T \phi_T h_d \frac{\Phi^q}{\Lambda'^q} \frac{1}{\Lambda} \\
&+ y_1^\tau \tau^c l \phi_T \phi_T h_d \frac{1}{\Lambda}.
\end{aligned} \tag{494}
$$

In our notation, all y with some subscripts denote Yukawa couplings of order 1 and Λ denotes a cut off scale of the A_4 symmetry. In order to obtain the natural hierarchy among lepton masses m_e, m_μ and m_τ, the Froggatt-Nielsen mechanism [228] is introduced as an additional $U(1)_{FN}$ flavor symmetry under which only the right-handed lepton sector is charged. Λ' is a cut off scale of the $U(1)_{FN}$ symmetry and Φ denotes the Froggatt-Nielsen flavon in Table 67. The $U(1)_{FN}$ charge values are taken as $2q$, q and 0 for e^c, μ^c and τ^c, respectively. By assuming that the flavon, carrying a negative unit charge of $U(1)_{FN}$, acquires a VEV $\langle\Phi\rangle/\Lambda' \equiv \lambda \ll 1$, the following mass ratio is realized through the Froggatt-Nielsen charges,

$$
m_e : m_\mu : m_\tau = \lambda^{2q} : \lambda^q : 1. \tag{495}
$$

If we take $q = 2$, the value $\lambda \sim 0.2$ is required to be consistent with the observed charged lepton mass hierarchy. The $U(1)_{FN}$ charges are listed in the fourth row of Table 67.

The superpotential associated with the Dirac neutrino mass is given as

$$
w_D = y_0^D \nu^c l h_u + y_1^D \nu^c l h_u \phi_T \frac{1}{\Lambda}, \tag{496}
$$

and for the right-handed Majorana sector, the superpotential is given as

$$
w_N = y_0^N \nu^c \nu^c \phi_S + y_1^N \nu^c \nu^c \xi + y_2^N \nu^c \nu^c \phi_T \xi \frac{1}{\Lambda} + y_3^N \nu^c \nu^c \phi_T \phi_S \frac{1}{\Lambda}, \tag{497}
$$

where there appear products of A_4 triplets such as $\mathbf{3} \times \mathbf{3} \times \mathbf{3}$ and $\mathbf{3} \times \mathbf{3} \times \mathbf{3} \times \mathbf{3}$.

The A_4 symmetry is spontaneously broken by VEVs of flavons. The tri-bimaximal mixing requires vacuum alignments of A_4 triplets ϕ_T and ϕ_S as follows:

$$
\langle(\phi_{T_1}, \phi_{T_2}, \phi_{T_3})\rangle = v_T(1,0,0), \qquad \langle(\phi_{S_1}, \phi_{S_2}, \phi_{S_3})\rangle = v_S(1,1,1). \tag{498}
$$

These vacuum alignments are realized in the scalar potential of the leading order [96, 97].

We write other VEVs as follows:

$$
\langle h_u \rangle = v_u, \qquad \langle h_d \rangle = v_d, \qquad \langle \xi \rangle = u. \tag{499}
$$

By inserting these VEVs in the superpotential of the charged lepton sector in Eq.(522), we obtain the charged lepton mass matrix M_E as

$$
M_E = \alpha_T v_d \begin{pmatrix} y_0^e \lambda^{2q} & 0 & 0 \\ 0 & y_0^\mu \lambda^q & 0 \\ 0 & 0 & y_0^\tau \end{pmatrix}, \tag{500}
$$

with

$$\alpha_T = \frac{v_T}{\Lambda}. \tag{501}$$

Since we have

$$m_e^2 = y_0^{e2}\lambda^{4q}\alpha_T^2 v_d^2, \qquad m_\mu^2 = y_0^{\mu 2}\lambda^{2q}\alpha_T^2 v_d^2, \qquad m_\tau^2 = y_0^{\tau 2}\alpha_T^2 v_d^2, \tag{502}$$

we can determine α_T from the tau lepton mass by fixing y_0^τ:

$$\alpha_T = \sqrt{\frac{m_\tau^2}{y_0^{\tau 2} v_d^2}}. \tag{503}$$

Now, we present the Dirac neutrino mass matrix in the leading order as follows:

$$M_D = v_u \begin{pmatrix} y_0^D & 0 & 0 \\ 0 & y_0^D & 0 \\ 0 & 0 & y_0^D \end{pmatrix}. \tag{504}$$

On the other hand, the right-handed Majorana mass matrix is given as

$$M_N = 2\Lambda \begin{pmatrix} \frac{2}{3}y_0^N\alpha_S + y_1^N\alpha_V & -\frac{1}{3}y_0^N\alpha_S & -\frac{1}{3}y_0^N\alpha_S \\ -\frac{1}{3}y_0^N\alpha_S & \frac{2}{3}y_0^N\alpha_S & -\frac{1}{3}y_0^N\alpha_S + y_1^N\alpha_V \\ -\frac{1}{3}y_0^N\alpha_S & -\frac{1}{3}y_0^N\alpha_S + y_1^N\alpha_V & \frac{2}{3}y_0^N\alpha_S \end{pmatrix}, \tag{505}$$

where

$$\alpha_S = \frac{v_S}{\Lambda}, \qquad \alpha_V = \frac{u}{\Lambda}. \tag{506}$$

By the seesaw mechanism $M_D^T M_R^{-1} M_D$, we get the neutrino mass matrix M_ν, which is rather complicated. We only display leading matrix elements which correspond to the neutrino mass matrix in Ref. [97]:

$$\begin{aligned} M_\nu &= \frac{1}{3}\begin{pmatrix} A+2B & A-B & A-B \\ A-B & A+\frac{1}{2}B+\frac{3}{2}C & A+\frac{1}{2}B-\frac{3}{2}C \\ A-B & a+\frac{1}{2}B-\frac{3}{2}C & A+\frac{1}{2}B+\frac{3}{2}C \end{pmatrix} \\ &= \frac{B+C}{2}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} + \frac{A-B}{3}\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} + \frac{B-C}{2}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \end{aligned} \tag{507}$$

where

$$A = k_0(y_0^{N^2}\alpha_S^2 - y_1^{N^2}\alpha_V^2), \quad B = k_0(y_0^N y_1^N \alpha_S\alpha_V - y_1^{N^2}\alpha_V^2),$$
$$C = k_0(y_0^N y_1^N \alpha_S\alpha_V + y_1^{N^2}\alpha_V^2), \quad k_0 = \frac{y_0^{D^2} v_u^2}{(y_0^{N^2} y_1^N \alpha_V\alpha_S^2 - y_1^{N^3}\alpha_V^3)\Lambda}.$$

At the leading order, neutrino masses are given as $m_1 = B$, $m_2 = A$, and $m_3 = C$. Our neutrino mass matrix is diagonalized by the tri-bimaximal mixing matrix $U_{\rm tri}$ in Eq.(493).

Let us estimate magnitudes of α_S and α_V to justify this model. The mass squared differences $\Delta m^2_{\rm atm} = \Delta m^2_{31}$ and $\Delta m^2_{\rm sol} = \Delta m^2_{12}$ are given as

$$\Delta m^2_{\rm atm} \simeq \pm \frac{(y_0^D v_u)^4}{\Lambda^2} \frac{y_0^N y_1^N \alpha_S \alpha_V}{[(y_0^N \alpha_S)^2 - (y_1^N \alpha_V)^2]^2},$$
$$\Delta m^2_{\rm sol} \simeq \frac{(y_0^N v_u)^4}{4\Lambda^2} \frac{y_0^N \alpha_S (y_0^N \alpha_S + 2 y_1^N \alpha_V)}{(y_1^N \alpha_V)^2 (y_0^N \alpha_S + y_1^N \alpha_V)^2}, \tag{508}$$

where the sign $+(-)$ in $\Delta m^2_{\rm atm}$ corresponds to the normal (inverted) mass hierarchy. We can obtain α_S and α_V from these equations. In the case of the normal mass hierarchy, putting

$$\alpha_S = k\ \alpha_V \quad (k > 0)\ , \tag{509}$$

we have

$$\Delta m^2_{\rm atm} \simeq \frac{(y_0^D v_u)^4}{\alpha_V^2 \Lambda^2} \frac{y_0^N y_1^N k}{(y_0^N k + y_1^N)^2 (y_0^N k - y_1^N)^2}, \qquad \Delta m^2_{\rm sol} \simeq \frac{(y_0^D v_u)^4}{4\alpha_V^2 \Lambda^2} \frac{y_0^N k (y_0^N k + 2 y_1^N)}{{y_1^N}^2 (y_0^N k + y_1^N)^2}. \tag{510}$$

The ratio of $\Delta m^2_{\rm atm}$ and $\Delta m^2_{\rm sol}$ is expressed in terms of k and Yukawa couplings as

$$\frac{\Delta m^2_{\rm atm}}{\Delta m^2_{\rm sol}} \simeq \frac{4 (y_1^N)^3}{(y_0^N k + 2 y_1^N)(y_0^N k - y_1^N)^2}\ . \tag{511}$$

Yukawa couplings are expected to be order one since there is no symmetry to suppress them. Then, by using Eq.(511), we get

$$k \simeq 1 \pm \frac{2}{\sqrt{3}} \sqrt{\frac{\Delta m^2_{\rm sol}}{\Delta m^2_{\rm atm}}} \simeq 1.2,\ \text{or}\ 0.8\ . \tag{512}$$

Thus, k is also expected to be order one, that is to say, $\alpha_S \sim \alpha_V$, which indicates that symmetry breaking scales of ξ and ϕ_S are the same order in the neutrino sector.

We also obtain a typical value:

$$\alpha_V \sim 5.8 \times 10^{-4}, \tag{513}$$

where we put $\Lambda = 2.4 \times 10^{18}$GeV, $\Delta m^2_{\rm atm} \sim 2.4 \times 10^{-3}{\rm eV}^2$, $\Delta m^2_{\rm sol} \sim 8.0 \times 10^{-5}{\rm eV}^2$ and $v_u = 165$GeV. In the following numerical calculations, we take magnitudes of Yukawa couplings to be $0.1 \sim 1$. It is found that α_V is lower than 10^{-3}, which is much smaller than $\alpha_T \simeq 0.032$ in the charged lepton sector.

In the case of the inverted mass hierarchy, the situation is different from the case of the normal one. As seen in $\Delta m^2_{\rm atm}$ of Eq.(508), the sign of y_0^N is opposite against y_1^N.

Therefore, the value of $(y_0^N \alpha_S + 2y_1^N \alpha_V)$ should be suppressed compared with $(y_1^N \alpha_V)$ in order to be consistent with the observed ratio $\Delta m^2_{\rm atm}/\Delta m^2_{\rm sol}$. In terms of the ratio r

$$r = \frac{y_1^N \alpha_V}{y_0^N \alpha_S + 2y_1^N \alpha_V} \ , \tag{514}$$

we have

$$\frac{\Delta m^2_{\rm atm}}{\Delta m^2_{\rm sol}} = -r \frac{(y_1^N \alpha_V)^2}{(y_0^N \alpha_S - y_1^N \alpha_V)^2} \ . \tag{515}$$

Therefore, we expect $r \sim -100$ for $y_0^N \alpha_S \sim -2y_1^N \alpha_V$. Then, we obtain a typical value:

$$\alpha_V \sim 1.1 \times 10^{-4}, \tag{516}$$

which is smaller than the one in the normal hierarchical case of Eq.(513).

Thus, the next leading orders $\mathcal{O}(\alpha_S^2)$ and $\mathcal{O}(\alpha_V^2)$ are neglected in the lepton mass matrices, and so the tri-bimaximal flavor mixing is justified in the model.

15.3 S_4 flavor model

The flavor symmetry is expected to explain the mass spectrum and the mixing matrix of both quarks and leptons. The tri-bimaximal mixing of leptons has been understood based on the non-Abelian finite group A_4 as presented in the previous subsection.

On the other hand, much attention has been devoted to the question whether these models can be extended to describe the observed pattern of quark masses and mixing angles, and whether these can be made compatible with the $SU(5)$ or $SO(10)$ grand unified theory (GUT). The attractive candidate is the S_4 symmetry, which has been already used for the neutrino masses and mixing [66, 67, 101]. The exact tri-bimaximal neutrino mixing is realized in the S_4 flavor model [70, 71, 72, 73, 74, 75]. The S_4 flavor models have been discussed for the lepton sector [76]-[85]. Although an attempt to unify the quark and lepton sectors was presented towards a grand unified theory of flavor [77, 78, 79], quark mixing angles are not predicted clearly.

We present a S_4 flavor model to unify the quarks and leptons in the framework of the $SU(5)$ GUT [86]. The S_4 group has 24 distinct elements and has five irreducible representations $\mathbf{1}$, $\mathbf{1'}$, $\mathbf{2}$, $\mathbf{3}$, and $\mathbf{3'}$. Three generations of $\overline{\mathbf{5}}$-plets in $SU(5)$ are assigned to $\mathbf{3}$ of S_4 while the first and the second generations of $\mathbf{10}$-plets in $SU(5)$ are assigned to $\mathbf{2}$ of S_4, and the third generation of $\mathbf{10}$-plet is to $\mathbf{1}$ of S_4. These assignments of S_4 for $\overline{\mathbf{5}}$ and $\mathbf{10}$ lead to the completely different structure of quark and lepton mass matrices. Right-handed neutrinos, which are $SU(5)$ gauge singlets, are also assigned to $\mathbf{2}$ for the first and second generations, and $\mathbf{1'}$ for the third generation, respectively. These assignments are essential to realize the tri-bimaximal mixing of neutrino flavors. Assignments of $SU(5)$, S_4, Z_4 and $U(1)_{FN}$ representations are summarized in Table 68, where ℓ, m and n are positive integers with the condition $m < n \leq 2m$. Taking vacuum alignments of relevant gauge singlet scalars, we predict the quark mixing as well as the tri-bimaximal mixing of leptons. Especially, the Cabbibo angle is predicted to be around 15° under the relevant vacuum alignments.

We present the S_4 flavor model in the framework of $SU(5)$ SUSY GUT. The flavor symmetry of quarks and leptons is the discrete group S_4 in our model. The group S_4 has irreducible representations **1**, **1′**, **2**, **3**, and **3′**.

	(T_1, T_2)	T_3	(F_1, F_2, F_3)	(N_e^c, N_μ^c)	N_τ^c	H_5	$H_{\bar{5}}$	H_{45}	Θ
$SU(5)$	10	10	$\bar{5}$	1	1	5	$\bar{5}$	45	1
S_4	**2**	**1**	**3**	**2**	**1′**	**1**	**1**	**1**	**1**
Z_4	$-i$	-1	i	1	1	1	1	-1	1
$U(1)_{FN}$	ℓ	0	0	m	0	0	0	0	-1

	(χ_1, χ_2)	(χ_3, χ_4)	(χ_5, χ_6, χ_7)	$(\chi_8, \chi_9, \chi_{10})$	$(\chi_{11}, \chi_{12}, \chi_{13})$	χ_{14}
$SU(5)$	1	1	1	1	1	1
S_4	**2**	**2**	**3′**	**3**	**3**	**1**
Z_4	$-i$	1	$-i$	-1	i	i
$U(1)_{FN}$	$-\ell$	$-n$	0	0	0	$-\ell$

Table 68: Assignments of $SU(5)$, S_4, Z_4, and $U(1)_{FN}$ representations.

Let us present the model of the quark and lepton flavor with the S_4 group in $SU(5)$ GUT. In $SU(5)$, matter fields are unified into $10(q_1, u^c, e^c)_L$ and $\bar{5}(d^c, l_e)_L$ dimensional representations. Three generations of $\bar{5}$, which are denoted by F_i, are assigned to **3** of S_4. On the other hand, the third generation of the 10-dimensional representation is assigned to **1** of S_4, so that the top quark Yukawa coupling is allowed in tree level. While, the first and the second generations are assigned to **2** of S_4. These 10-dimensional representations are denoted by T_3 and (T_1, T_2), respectively. Right-handed neutrinos, which are $SU(5)$ gauge singlets, are also assigned to **1′** and **2** for N_τ^c and (N_e^c, N_μ^c), respectively.

We introduce new scalars χ_i in addition to the 5-dimensional, $\bar{5}$-dimensional and 45-dimensional Higgs of the $SU(5)$, H_5, $H_{\bar{5}}$, and H_{45} which are assigned to **1** of S_4. These new scalars are supposed to be $SU(5)$ gauge singlets. The (χ_1, χ_2) and (χ_3, χ_4) scalars are assigned to **2**, (χ_5, χ_6, χ_7) are assigned to **3′**, $(\chi_8, \chi_9, \chi_{10})$ and $(\chi_{11}, \chi_{12}, \chi_{13})$ are **3**, and χ_{14} is assigned to **1** of the S_4 representations, respectively. In the leading order, (χ_3, χ_4) are coupled with the right-handed Majorana neutrino sector, (χ_5, χ_6, χ_7) are coupled with the Dirac neutrino sector, $(\chi_8, \chi_9, \chi_{10})$ and $(\chi_{11}, \chi_{12}, \chi_{13})$ are coupled with the charged lepton and down-type quark sectors, respectively. In the next-leading order, (χ_1, χ_2) scalars are coupled with the up-type quark sector, and the S_4 singlet χ_{14} contributes to the charged lepton and down-type quark sectors, and then the mass ratio of the electron and the down quark is reproduced properly. We also add Z_4 symmetry in order to obtain relevant couplings. In order to obtain the natural hierarchy among quark and lepton masses, the Froggatt-Nielsen mechanism [228] is introduced as an additional $U(1)_{FN}$ flavor symmetry. Θ denotes the Froggatt-Nielsen flavon. The particle assignments of $SU(5)$, S_4 and Z_4 and $U(1)_{FN}$ are summarized Table 68. The $U(1)_{FN}$ charges ℓ, m and n will be determined phenomenologically.

We can now write down the superpotential respecting S_4, Z_4 and $U(1)_{FN}$ symmetries in terms of the S_4 cutoff scale Λ, and the $U(1)_{FN}$ cutoff scale $\overline{\Lambda}$. The $SU(5)$ invariant superpotential of the Yukawa sector up to the linear terms of χ_i is given as

$$\begin{aligned}
w_{SU(5)} &= y_1^u(T_1,T_2)\otimes T_3\otimes(\chi_1,\chi_2)\otimes H_5/\Lambda + y_2^u T_3\otimes T_3\otimes H_5 \\
&+ y_1^N(N_e^c,N_\mu^c)\otimes(N_e^c,N_\mu^c)\otimes\Theta^{2m}/\bar{\Lambda}^{2m-1} \\
&+ y_2^N(N_e^c,N_\mu^c)\otimes(N_e^c,N_\mu^c)\otimes(\chi_3,\chi_4)\otimes\Theta^{2m-n}/\bar{\Lambda}^{2m-n} + MN_\tau^c\otimes N_\tau^c \\
&+ y_1^D(N_e^c,N_\mu^c)\otimes(F_1,F_2,F_3)\otimes(\chi_5,\chi_6,\chi_7)\otimes H_5\otimes\Theta^m/(\Lambda\bar{\Lambda}^m) \\
&+ y_2^D N_\tau^c\otimes(F_1,F_2,F_3)\otimes(\chi_5,\chi_6,\chi_7)\otimes H_5/\Lambda \\
&+ y_1(F_1,F_2,F_3)\otimes(T_1,T_2)\otimes(\chi_8,\chi_9,\chi_{10})\otimes H_{45}\otimes\Theta^\ell/(\Lambda\bar{\Lambda}^\ell) \\
&+ y_2(F_1,F_2,F_3)\otimes T_3\otimes(\chi_{11},\chi_{12},\chi_{13})\otimes H_{\bar{5}}/\Lambda,
\end{aligned} \tag{517}$$

where y_1^u, y_2^u, y_1^N, y_2^N, y_1^D, y_2^D, y_1, and y_2 are Yukawa couplings. The $U(1)_{FN}$ charges ℓ, m, and n are integers, and satisfy the conditions $m-n<0$, $2m-n\geq 0$. In our numerical study, we take $\ell=m=1$ and $n=2$. Then, some couplings are forbidden in the superpotential. We discuss the feature of the quark and lepton mass matrices and flavor mixing based on this superpotential. However, we will take into account the next leading couplings as to χ_i in the numerical study of the flavor mixing and CP violation.

We begin to discuss the lepton sector of the superpotential $w^{(0)}_{SU(5)}$. Denoting Higgs doublets as h_u and h_d, the superpotential of the Yukawa sector respecting the $S_4\times Z_4\times U(1)_{FN}$ symmetry is given for charged leptons as

$$\begin{aligned}
w_l = &-3y_1\left[\frac{e^c}{\sqrt{2}}(l_\mu\chi_9 - l_\tau\chi_{10}) + \frac{\mu^c}{\sqrt{6}}(-2l_e\chi_8 + l_\mu\chi_9 + l_\tau\chi_{10})\right]h_{45}\Theta^\ell/(\Lambda\bar{\Lambda}^\ell) \\
&+ y_2\tau^c(l_e\chi_{11}+l_\mu\chi_{12}+l_\tau\chi_{13})h_d/\Lambda.
\end{aligned} \tag{518}$$

For right-handed Majorana neutrinos, the superpotential is given as

$$\begin{aligned}
w_N &= y_1^N(N_e^cN_e^c + N_\mu^cN_\mu^c)\Theta^{2m}/\bar{\Lambda}^{2m-1} \\
&+ y_2^N\left[(N_e^cN_\mu^c + N_\mu^cN_e^c)\chi_3 + (N_e^cN_e^c - N_\mu^cN_\mu^c)\chi_4\right]\Theta^{2m-n}/\bar{\Lambda}^{2m-n} + MN_\tau^cN_\tau^c,
\end{aligned} \tag{519}$$

and for Dirac neutrino Yukawa couplings, the superpotential is

$$\begin{aligned}
w_D = &\ y_1^D\left[\frac{N_e^c}{\sqrt{6}}(2l_e\chi_5 - l_\mu\chi_6 - l_\tau\chi_7) + \frac{N_\mu^c}{\sqrt{2}}(l_\mu\chi_6 - l_\tau\chi_7)\right]h_u\Theta^m/(\Lambda\bar{\Lambda}^m) \\
&+ y_2^D N_\tau^c(l_e\chi_5 + l_\mu\chi_6 + l_\tau\chi_7)h_u/\Lambda.
\end{aligned} \tag{520}$$

Higgs doublets h_u, h_d and gauge singlet scalars Θ and χ_i, are assumed to develop their VEVs as follows:

$$\begin{aligned}
&\langle h_u\rangle = v_u, \quad \langle h_d\rangle = v_d, \quad \langle h_{45}\rangle = v_{45}, \quad \langle\Theta\rangle = \theta, \\
&\langle(\chi_3,\chi_4)\rangle = (u_3,u_4), \quad \langle(\chi_5,\chi_6,\chi_7)\rangle = (u_5,u_6,u_7), \\
&\langle(\chi_8,\chi_9,\chi_{10})\rangle = (u_8,u_9,u_{10}), \quad \langle(\chi_{11},\chi_{12},\chi_{13})\rangle = (u_{11},u_{12},u_{13}),
\end{aligned} \tag{521}$$

which are supposed to be real. Then, we obtain the mass matrix for charged leptons as

$$M_l = -3y_1\lambda^\ell v_{45}\begin{pmatrix}0 & \alpha_9/\sqrt{2} & -\alpha_{10}/\sqrt{2}\\ -2\alpha_8/\sqrt{6} & \alpha_9/\sqrt{6} & \alpha_{10}/\sqrt{6}\\ 0 & 0 & 0\end{pmatrix} + y_2 v_d\begin{pmatrix}0 & 0 & 0\\ 0 & 0 & 0\\ \alpha_{11} & \alpha_{12} & \alpha_{13}\end{pmatrix}, \tag{522}$$

while the right-handed Majorana neutrino mass matrix is given as

$$M_N = \begin{pmatrix}\lambda^{2m-n}(y_1^N\lambda^n\bar{\Lambda} + y_2^N\alpha_4\Lambda) & y_2^N\lambda^{2m-n}\alpha_3\Lambda & 0\\ y_2^N\lambda^{2m-n}\alpha_3\Lambda & \lambda^{2m-n}(y_1^N\lambda^n\bar{\Lambda} - y_2^N\alpha_4\Lambda) & 0\\ 0 & 0 & M\end{pmatrix}. \tag{523}$$

Because of the condition $m-n<0$, $(1,3)$, $(2,3)$, $(3,1)$ and $(3,3)$ elements of the right-handed Majorana neutrino mass matrix vanish. These are so called SUSY zeros. The Dirac mass matrix of neutrinos is

$$M_D = y_1^D\lambda^m v_u\begin{pmatrix}2\alpha_5/\sqrt{6} & -\alpha_6/\sqrt{6} & -\alpha_7/\sqrt{6}\\ 0 & \alpha_6/\sqrt{2} & -\alpha_7/\sqrt{2}\\ 0 & 0 & 0\end{pmatrix} + y_2^D v_u\begin{pmatrix}0 & 0 & 0\\ 0 & 0 & 0\\ \alpha_5 & \alpha_6 & \alpha_7\end{pmatrix}, \tag{524}$$

where we denote $\alpha_i \equiv u_i/\Lambda$ and $\lambda \equiv \theta/\bar{\Lambda}$.

In order to get the left-handed mixing of charged leptons, we investigate $M_l^\dagger M_l$. If we can take vacuum alignment $(u_8, u_9, u_{10}) = (0, u_9, 0)$ and $(u_{11}, u_{12}, u_{13}) = (0, 0, u_{13})$, that is $\alpha_8 = \alpha_{10} = \alpha_{11} = \alpha_{12} = 0$, we obtain

$$M_l = \begin{pmatrix}0 & -3y_1\lambda^\ell\alpha_9 v_{45}/\sqrt{2} & 0\\ 0 & -3y_1\lambda^\ell\alpha_9 v_{45}/\sqrt{6} & 0\\ 0 & 0 & y_2\alpha_{13}v_d\end{pmatrix}, \tag{525}$$

then $M_l^\dagger M_l$ is as follows:

$$M_l^\dagger M_l = v_d^2\begin{pmatrix}0 & 0 & 0\\ 0 & 6|\bar{y}_1\lambda^\ell\alpha_9|^2 & 0\\ 0 & 0 & |y_2|^2\alpha_{13}^2\end{pmatrix}, \tag{526}$$

where we replace $y_1 v_{45}$ with $\bar{y}_1 v_d$. We find $\theta_{12}^l = \theta_{13}^l = \theta_{23}^l = 0$, where θ_{ij}^l denote left-handed mixing angles to diagonalize the charged lepton mass matrix. Then, charged lepton masses are

$$m_e^2 = 0\ , \quad m_\mu^2 = 6|\bar{y}_1\lambda^\ell\alpha_9|^2 v_d^2\ , \quad m_\tau^2 = |y_2|^2\alpha_{13}^2 v_d^2\ . \tag{527}$$

It is remarkable that the electron mass vanishes. We will discuss the electron mass in the next leading order.

Taking vacuum alignment $(u_3, u_4) = (0, u_4)$ and $(u_5, u_6, u_7) = (u_5, u_5, u_5)$ in Eq.(523), the right-handed Majorana mass matrix of neutrinos turns to

$$M_N = \begin{pmatrix}\lambda^{2m-n}(y_1^N\lambda^n\bar{\Lambda} + y_2^N\alpha_4\Lambda) & 0 & 0\\ 0 & \lambda^{2m-n}(y_1^N\lambda^n\bar{\Lambda} - y_2^N\alpha_4\Lambda) & 0\\ 0 & 0 & M\end{pmatrix}, \tag{528}$$

and the Dirac mass matrix of neutrinos turns to

$$M_D = y_1^D \lambda^m v_u \begin{pmatrix} 2\alpha_5/\sqrt{6} & -\alpha_5/\sqrt{6} & -\alpha_5/\sqrt{6} \\ 0 & \alpha_5/\sqrt{2} & -\alpha_5/\sqrt{2} \\ 0 & 0 & 0 \end{pmatrix} + y_2^D v_u \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \alpha_5 & \alpha_5 & \alpha_5 \end{pmatrix} . \tag{529}$$

By using the seesaw mechanism $M_\nu = M_D^T M_N^{-1} M_D$, the left-handed Majorana neutrino mass matrix is written as

$$M_\nu = \begin{pmatrix} a + \frac{2}{3}b & a - \frac{1}{3}b & a - \frac{1}{3}b \\ a - \frac{1}{3}b & a + \frac{1}{6}b + \frac{1}{2}c & a + \frac{1}{6}b - \frac{1}{2}c \\ a - \frac{1}{3}b & a + \frac{1}{6}b - \frac{1}{2}c & a + \frac{1}{6}b + \frac{1}{2}c \end{pmatrix} , \tag{530}$$

where

$$a = \frac{(y_2^D \alpha_5 v_u)^2}{M}, \qquad b = \frac{(y_1^D \alpha_5 v_u \lambda^m)^2}{\lambda^{2m-n}(y_1^N \lambda^n \bar{\Lambda} + y_2^N \alpha_4 \Lambda)}, \qquad c = \frac{(y_1^D \alpha_5 v_u \lambda^m)^2}{\lambda^{2m-n}(y_1^N \lambda^n \bar{\Lambda} - y_2^N \alpha_4 \Lambda)}. \tag{531}$$

The neutrino mass matrix is decomposed as

$$M_\nu = \frac{b+c}{2} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} + \frac{3a-b}{3} \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} + \frac{b-c}{2} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} , \tag{532}$$

which gives the tri-bimaximal mixing matrix $U_{\text{tri-bi}}$ and mass eigenvalues as follows:

$$U_{\text{tri-bi}} = \begin{pmatrix} \frac{2}{\sqrt{6}} & \frac{1}{\sqrt{3}} & 0 \\ -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} \end{pmatrix} , \qquad m_1 = b \ , \qquad m_2 = 3a \ , \qquad m_3 = c \ . \tag{533}$$

The next leading terms of the superpotential are important to predict the deviation from the tri-bimaximal mixing of leptons, especially, U_{e3}. The relevant superpotential in the charged lepton sector is given at the next leading order as

$$\begin{aligned} \Delta w = & \, y_{\Delta_a}(T_1, T_2) \otimes (F_1, F_2, F_3) \otimes (\chi_1, \chi_2) \otimes (\chi_{11}, \chi_{12}, \chi_{13}) \otimes H_{\bar{5}}/\Lambda^2 \\ & + y_{\Delta_b}(T_1, T_2) \otimes (F_1, F_2, F_3) \otimes (\chi_5, \chi_6, \chi_7) \otimes \chi_{14} \otimes H_{\bar{5}}/\Lambda^2 \\ & + y_{\Delta_c}(T_1, T_2) \otimes (F_1, F_2, F_3) \otimes (\chi_1, \chi_2) \otimes (\chi_5, \chi_6, \chi_7) \otimes H_{45}/\Lambda^2 \\ & + y_{\Delta_d}(T_1, T_2) \otimes (F_1, F_2, F_3) \otimes (\chi_{11}, \chi_{12}, \chi_{13}) \otimes \chi_{14} \otimes H_{45}/\Lambda^2 \\ & + y_{\Delta_e} T_3 \otimes (F_1, F_2, F_3) \otimes (\chi_5, \chi_6, \chi_7) \otimes (\chi_8, \chi_9, \chi_{10}) \otimes H_{\bar{5}} \otimes /\Lambda^2 \\ & + y_{\Delta_f} T_3 \otimes (F_1, F_2, F_3) \otimes (\chi_8, \chi_9, \chi_{10}) \otimes (\chi_{11}, \chi_{12}, \chi_{13}) \otimes H_{45} \otimes /\Lambda^2 \ . \end{aligned} \tag{534}$$

In order to estimate the effect of this superpotential on the lepton flavor mixing, we calculate the next leading terms of the charged lepton mass matrix elements ϵ_{ij}, which are given as

$$\begin{aligned}
\epsilon_{11} &= y_{\Delta_b}\alpha_5\alpha_{14}v_d - 3\bar{y}_{\Delta_{c2}}\alpha_1\alpha_5 v_d, \\
\epsilon_{12} &= -\frac{1}{2}y_{\Delta_b}\alpha_5\alpha_{14}v_d + 3\left[\frac{\sqrt{3}}{4}(\sqrt{3}-1)\bar{y}_{\Delta_{c1}} - \frac{1}{4}(\sqrt{3}+1)\bar{y}_{\Delta_{c2}}\right]\alpha_1\alpha_5 v_d, \\
\epsilon_{13} &= \left[\left\{\frac{\sqrt{3}}{4}(\sqrt{3}-1)y_{\Delta_{a1}} + \frac{1}{4}(\sqrt{3}+1)y_{\Delta_{a2}}\right\}\alpha_1\alpha_{13} - \frac{1}{2}y_{\Delta_b}\alpha_5\alpha_{14}\right]v_d, \\
&\quad - 3\left[\left\{-\frac{\sqrt{3}}{4}(\sqrt{3}+1)\bar{y}_{\Delta_{c1}} - \frac{1}{4}(\sqrt{3}-1)\bar{y}_{\Delta_{c2}}\right\}\alpha_1\alpha_5 + \frac{\sqrt{3}}{2}\bar{y}_{\Delta_d}\alpha_{13}\alpha_{14}\right]v_d, \\
\epsilon_{21} &= -3\bar{y}_{\Delta_{c1}}\alpha_1\alpha_5 v_d, \\
\epsilon_{22} &= \frac{\sqrt{3}}{2}y_{\Delta_b}\alpha_5\alpha_{14}v_d + 3\left[\frac{1}{4}(\sqrt{3}-1)\bar{y}_{\Delta_{c1}} + \frac{\sqrt{3}}{4}(\sqrt{3}+1)\bar{y}_{\Delta_{c2}}\right]\alpha_1\alpha_5 v_d, \\
\epsilon_{23} &= \left[\left\{-\frac{1}{4}(\sqrt{3}-1)y_{\Delta_{a1}} + \frac{\sqrt{3}}{4}(\sqrt{3}+1)y_{\Delta_{a2}}\right\}\alpha_1\alpha_{13} - \frac{\sqrt{3}}{2}y_{\Delta_b}\alpha_5\alpha_{14}\right]v_d, \\
&\quad - 3\left[\left\{\frac{1}{4}(\sqrt{3}+1)\bar{y}_{\Delta_{c1}} - \frac{\sqrt{3}}{4}(\sqrt{3}-1)\bar{y}_{\Delta_{c2}}\right\}\alpha_1\alpha_5 - \frac{1}{2}\bar{y}_{\Delta_d}\alpha_{13}\alpha_{14}\right]v_d, \\
\epsilon_{31} &= -y_{\Delta_e}\alpha_5\alpha_9 v_d - 3\bar{y}_{\Delta_f}\alpha_9\alpha_{13}v_d, \\
\epsilon_{32} &= 0, \\
\epsilon_{33} &= y_{\Delta_e}\alpha_5\alpha_9 v_d.
\end{aligned} \tag{535}$$

Magnitudes of ϵ_{ij}'s are of $\mathcal{O}(\tilde{\alpha}^2)$, where $\tilde{\alpha}$ is a linear combination of α_i's. The charged lepton mass matrix is written in terms of ϵ_{ij} as

$$M_l \simeq \begin{pmatrix} \epsilon_{11} & \frac{\sqrt{3}m_\mu}{2} + \epsilon_{12} & \epsilon_{13} \\ \epsilon_{21} & \frac{m_\mu}{2} + \epsilon_{22} & \epsilon_{23} \\ \epsilon_{31} & \epsilon_{32} & m_\tau + \epsilon_{33} \end{pmatrix}, \tag{536}$$

where m_μ and m_τ are given in Eq.(527). Then, $M_l^\dagger M_l$ is not diagonal due to next leading terms ϵ_{ij}, which give the non-vanishing electron mass. Since we have $m_\mu = \mathcal{O}(\lambda\tilde{\alpha})$ and $m_\tau = \mathcal{O}(\tilde{\alpha})$ as seen in Eq.(527), and $\epsilon_{ij} = \mathcal{O}(m_e)$, the left-handed charged lepton mixing matrix is written as

$$U_E = \begin{pmatrix} 1 & \mathcal{O}\left(\frac{m_e}{m_\mu}\right) & \mathcal{O}\left(\frac{m_e}{m_\tau}\right) \\ \mathcal{O}\left(\frac{m_e}{m_\mu}\right) & 1 & \mathcal{O}\left(\frac{m_e}{m_\tau}\right) \\ \mathcal{O}\left(\frac{m_e}{m_\tau}\right) & \mathcal{O}\left(\frac{m_e}{m_\tau}\right) & 1 \end{pmatrix}. \tag{537}$$

Now, the lepton mixing matrix $U_{\rm MNS}$ is deviated from the tri-bimaximal mixing as follows:

$$U_{\rm MNS} = U_E^\dagger U_{\text{tri-bi}}. \tag{538}$$

The lepton mixing matrix elements $U_{e3},\ U_{e2}.\ U_{\mu 3}$ are given as

$$U_{e3} \sim \frac{1}{\sqrt{2}}\left(\mathcal{O}\left(\frac{m_e}{m_\mu}\right)\right), \quad U_{e2} \sim \frac{1}{\sqrt{3}}\left(1+\mathcal{O}\left(\frac{m_e}{m_\mu}\right)\right), \quad U_{\mu 3} \sim -\frac{1}{\sqrt{2}}\left(1-\mathcal{O}\left(\frac{m_e}{m_\tau}\right)\right) . \tag{539}$$

Thus, the deviation from the tri-bimaximal mixing is lower than $\mathcal{O}\,(0.01)$.

The superpotential of the next leading order for Majorana neutrinos is

$$\begin{aligned}\Delta w^N_{SU(5)} = {} & y^N_{\Delta_1}(N^c_e, N^c_\mu) \otimes (N^c_e, N^c_\mu) \otimes (\chi_1, \chi_2) \otimes \chi_{14}/\Lambda \\ & + y^N_{\Delta_2}(N^c_e, N^c_\mu) \otimes N^c_\tau \otimes (\chi_5, \chi_6, \chi_7) \otimes (\chi_{11}, \chi_{12}, \chi_{13}) \otimes \Theta/(\Lambda\bar{\Lambda}) \\ & + y^N_{\Delta_3}(N^c_e, N^c_\mu) \otimes N^c_\tau \otimes (\chi_8, \chi_9, \chi_{10}) \otimes (\chi_8, \chi_9, \chi_{10}) \otimes \Theta/(\Lambda\bar{\Lambda}) \\ & + y^N_{\Delta_4} N^c_\tau \otimes N^c_\tau \otimes (\chi_8, \chi_9, \chi_{10}) \otimes (\chi_8, \chi_9, \chi_{10})/\Lambda. \end{aligned} \tag{540}$$

The dominant matrix elements of the Majorana neutrinos at the next leading order are given as follows:

$$\Delta M_N = \Lambda \times \begin{pmatrix} y^N_{\Delta_1}\alpha_1\alpha_{14} & y^N_{\Delta_1}\alpha_1\alpha_{14} & -\frac{\lambda}{\sqrt{6}}y^N_{\Delta_2}\alpha_5\alpha_{13} + \frac{\lambda}{\sqrt{2}}y^N_{\Delta_3}\lambda\alpha_9^2 \\ y^N_{\Delta_1}\alpha_1\alpha_{14} & -y^N_{\Delta_1}\alpha_1\alpha_{14} & -\frac{\lambda}{\sqrt{2}}y^N_{\Delta_2}\alpha_5\alpha_{13} + \frac{\lambda}{\sqrt{6}}y^N_{\Delta_3}\alpha_9^2 \\ -\frac{\lambda}{\sqrt{6}}y^N_{\Delta_2}\alpha_5\alpha_{13} + \frac{\lambda}{\sqrt{2}}y^N_{\Delta_3}\alpha_9^2 & -\frac{\lambda}{\sqrt{2}}y^N_{\Delta_2}\alpha_5\alpha_{13} + \frac{\lambda}{\sqrt{6}}y^N_{\Delta_3}\alpha_9^2 & y^N_{\Delta_4}\alpha_9^2 \end{pmatrix} . \tag{541}$$

Then the U_{e3} is estimated as follows:

$$U_{e3} \sim \frac{y^N_{\Delta_1}\alpha_1\alpha_{14}}{y^N_2\alpha_4} \sim \mathcal{O}(\tilde{\alpha}) . \tag{542}$$

We also consider the Dirac neutrino mass matrix. The superpotential at the next leading order for Dirac neutrino is given as

$$\Delta w^D_{SU(5)} = y^D_\Delta(N^c_e, N^c_\mu) \otimes (F_1, F_2, F_3) \otimes (\chi_8, \chi_9, \chi_{10}) \otimes (\chi_{11}, \chi_{12}, \chi_{13}) \otimes H_5 \otimes \Theta/(\Lambda^2\bar{\Lambda}) . \tag{543}$$

The dominat matrix elements of the Dirac neutrinos at the next leading order are given as follows:

$$\Delta M_D = \begin{pmatrix} * & * & * \\ y^D_\Delta \lambda\alpha_9\alpha_{13}v_u & * & * \\ * & * & *. \end{pmatrix} \tag{544}$$

Then, we can estimate U_{e3} as follows:

$$U_{e3} \sim -\frac{\sqrt{6}y^D_\Delta\alpha_9\alpha_{13}}{3y^D_1\alpha_5} \sim \mathcal{O}(\tilde{\alpha}) . \tag{545}$$

Thus, the contribution of the next leading terms on U_{e3} is of order $\mathcal{O}(\tilde{\alpha})$ in the neutrino sector while that is $\mathcal{O}(m_e/m_\mu)$ in the charged lepton sector. Therefore, it is concluded that the deviation from the tri-bimaximal mixing mainly comes from the neutrino sector.

Let us discuss the quark sector. For down-type quarks, we can write the superpotential as follows:

$$w_d = y_1 \left[\frac{1}{\sqrt{2}} (s^c \chi_9 - b^c \chi_{10}) q_1 + \frac{1}{\sqrt{6}} (-2 d^c \chi_8 + s^c \chi_9 + b^c \chi_{10}) q_2 \right] h_{45} \Theta^\ell / (\Lambda \bar{\Lambda}^\ell)$$
$$+ y_2 (d^c \chi_{11} + s^c \chi_{12} + b^c \chi_{13}) q_3 h_d / \Lambda. \tag{546}$$

Since the vacuum alignment is fixed in the lepton sector as seen in Eq.(521), the down-type quark mass matrix at the leading order is given as

$$M_d = v_d \begin{pmatrix} 0 & 0 & 0 \\ \bar{y}_1 \lambda^\ell \alpha_9 / \sqrt{2} & \bar{y}_1 \lambda^\ell \alpha_9 / \sqrt{6} & 0 \\ 0 & 0 & y_2 \alpha_{13} \end{pmatrix}, \tag{547}$$

where we denote $\bar{y}_1 v_d = y_1' v_{45}$. Then, we have

$$M_d^\dagger M_d = v_d^2 \begin{pmatrix} \frac{1}{2} |\bar{y}_1 \lambda^\ell \alpha_9|^2 & \frac{1}{2\sqrt{3}} |\bar{y}_1 \lambda^\ell \alpha_9|^2 & 0 \\ \frac{1}{2\sqrt{3}} |\bar{y}_1 \lambda^\ell \alpha_9|^2 & \frac{1}{6} |\bar{y}_1 \lambda^\ell \alpha_9|^2 & 0 \\ 0 & 0 & |y_2|^2 \alpha_{13}^2 \end{pmatrix}. \tag{548}$$

This matrix can be diagonalized by the orthogonal matrix U_d as

$$U_d = \begin{pmatrix} \cos 60^\circ & \sin 60^\circ & 0 \\ -\sin 60^\circ & \cos 60^\circ & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{549}$$

The down-type quark masses are given as

$$m_d^2 = 0\,, \quad m_s^2 = \frac{2}{3} |\bar{y}_1 \lambda^\ell \alpha_9|^2 v_d^2\,, \quad m_b^2 \approx |y_2|^2 \alpha_{13}^2 v_d^2\,, \tag{550}$$

which correspond to ones of charged lepton masses in Eq.(527). The down quark mass vanishes as well as the electron mass, however tiny masses appear in the next leading order.

The down-type quark mass matrix including the next leading order is

$$M_d \simeq \begin{pmatrix} \bar{\epsilon}_{11} & \bar{\epsilon}_{21} & \bar{\epsilon}_{31} \\ \frac{\sqrt{3} m_s}{2} + \bar{\epsilon}_{12} & \frac{m_s}{2} + \bar{\epsilon}_{22} & \bar{\epsilon}_{32} \\ \bar{\epsilon}_{13} & \bar{\epsilon}_{23} & m_b + \bar{\epsilon}_{33} \end{pmatrix}, \tag{551}$$

where $\bar{\epsilon}_{ij}$'s are given by replacing $\bar{y}_{\Delta_i}$ with $-1/3 \bar{y}_{\Delta_i}$ $(i = c1, c2, d, f)$ in Eq.(535), and m_s and m_b are given in Eq.(550).

By rotating $M_d^\dagger M_d$ with the mixing matrix U_d in Eq.(549), we have

$$U_d^\dagger M_d^\dagger M_d U_d \simeq \begin{pmatrix} |m_d|^2 & \mathcal{O}(\tilde{\alpha}^2 m_s) & \frac{1}{2} (\bar{\epsilon}_{13}^* - \sqrt{3} \bar{\epsilon}_{23}^*) m_b \\ \mathcal{O}(\tilde{\alpha}^2 m_s^*) & |m_s|^2 & \frac{1}{2} (\sqrt{3} \bar{\epsilon}_{13}^* + \bar{\epsilon}_{23}^*) m_b \\ \frac{1}{2} (\bar{\epsilon}_{13} - \sqrt{3} \bar{\epsilon}_{23}) m_b^* & \frac{1}{2} (\sqrt{3} \bar{\epsilon}_{13} + \bar{\epsilon}_{23}) m_b^* & |m_b|^2 \end{pmatrix}. \tag{552}$$

Then we get mixing angles $\theta^d_{12},\ \theta^d_{13},\ \theta^d_{23}$ in the mass matrix of Eq.(552) as

$$\theta^d_{12} = \mathcal{O}\left(\frac{m_d}{m_s}\right), \qquad \theta^d_{13} = \mathcal{O}\left(\frac{m_d}{m_b}\right), \qquad \theta^d_{23} = \mathcal{O}\left(\frac{m_d}{m_b}\right), \tag{553}$$

where a CP violating phase is neglected.

Let us discuss the up-type quark sector. The superpotential respecting $S_4 \times Z_4 \times U(1)_{FN}$ is given as

$$w_u = y^u_1 \left[(u^c\chi_1 + c^c\chi_2)q_3 + t^c(q_1\chi_1 + q_2\chi_2)\right] h_u/\Lambda + y^u_2 t^c q_3 h_u \ . \tag{554}$$

We denote their VEVs as follows:

$$\langle(\chi_1, \chi_2)\rangle = (u_1, u_2) \ . \tag{555}$$

Then, we obtain the mass matrix for up-type quarks as

$$M_u = v_u \begin{pmatrix} 0 & 0 & y^u_1\alpha_1 \\ 0 & 0 & y^u_1\alpha_2 \\ y^u_1\alpha_1 & y^u_1\alpha_2 & y^u_2 \end{pmatrix} . \tag{556}$$

The next leading terms of the superpotential are also important to predict the CP violation in the quark sector. The relevant superpotential is given at the next leading order as

$$\begin{aligned} \Delta w_u = {} & y^u_{\Delta_a}(T_1, T_2) \otimes (T_1, T_2) \otimes (\chi_1, \chi_2) \otimes (\chi_1, \chi_2) \otimes H_5/\Lambda^2 \\ & + y^u_{\Delta_b}(T_1, T_2) \otimes (T_1, T_2) \otimes \chi_{14} \otimes \chi_{14} \otimes H_5/\Lambda^2 \\ & + y^u_{\Delta_c} T_3 \otimes T_3 \otimes (\chi_8, \chi_9, \chi_{10}) \otimes (\chi_8, \chi_9, \chi_{10}) \otimes H_5/\Lambda^2. \end{aligned} \tag{557}$$

We have the following mass matrix, in which the next leading terms are added to the up-type quark mass matrix of Eq.(556):

$$M_u = v_u \begin{pmatrix} 2y^u_{\Delta_{a1}}\alpha_1^2 + y^u_{\Delta_b}\alpha_{14}^2 & y^u_{\Delta_{a2}}\alpha_1^2 & y^u_1\alpha_1 \\ y^u_{\Delta_{a2}}\alpha_1^2 & 2y^u_{\Delta_{a1}}\alpha_1^2 + y^u_{\Delta_b}\alpha_{14}^2 & y^u_1\alpha_1 \\ y^u_1\alpha_1 & y^u_1\alpha_1 & y^u_2 + y^u_{\Delta_c}\alpha_9^2 \end{pmatrix}, \tag{558}$$

where we take the alignment $\alpha_1 = \alpha_2$. After rotating the mass matrix M_u by $\theta_{12} = 45°$, we get

$$\hat{M}_u \approx v_u \begin{pmatrix} (2y^u_{\Delta_{a1}} - y^u_{\Delta_{a2}})\alpha_1^2 + y^u_{\Delta_b}\alpha_{14}^2 & 0 & 0 \\ 0 & (2y^u_{\Delta_{a1}} + y^u_{\Delta_{a2}})\alpha_1^2 + y^u_{\Delta_b}\alpha_{14}^2 & \sqrt{2}y^u_1\alpha_1 \\ 0 & \sqrt{2}y^u_1\alpha_1 & y^u_2 \end{pmatrix}. \tag{559}$$

This mass matrix is taken to be real one by removing phases. The matrix is diagonalized by the orthogonal transformation as $V_u^T \hat{M}_u V_{\rm F}$, where

$$V_u \simeq \begin{pmatrix} 1 & 0 & 0 \\ 0 & r_t & r_c \\ 0 & -r_c & r_t \end{pmatrix}, \quad r_c = \sqrt{\frac{m_c}{m_c + m_t}}, \quad \text{and} \quad r_t = \sqrt{\frac{m_t}{m_c + m_t}} . \tag{560}$$

Now we can discuss the CKM matrix. Mixing matrices of up- and down-type quarks are summarized as

$$U_u \simeq \begin{pmatrix} \cos 45^\circ & \sin 45^\circ & 0 \\ -\sin 45^\circ & \cos 45^\circ & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & e^{-i\rho} & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & r_t & r_c \\ 0 & -r_c & r_t \end{pmatrix},$$

$$U_d \simeq \begin{pmatrix} \cos 60^\circ & \sin 60^\circ & 0 \\ -\sin 60^\circ & \cos 60^\circ & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & \theta_{12}^d & \theta_{13}^d \\ -\theta_{12}^d - \theta_{13}^d\theta_{23}^d & 1 & \theta_{23}^d \\ -\theta_{13}^d + \theta_{12}^d\theta_{23}^d & -\theta_{23}^d - \theta_{12}^d\theta_{13}^d & 1 \end{pmatrix}. \quad (561)$$

Therefore, the CKM matrix can be written as

$$V^{CKM} = U_u^\dagger U_d \approx \begin{pmatrix} 1 & 0 & 0 \\ 0 & r_t & -r_c \\ 0 & r_c & r_t \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & e^{i\rho} & 0 \\ 0 & 0 & 1 \end{pmatrix}$$
$$\times \begin{pmatrix} \cos 15^\circ & \sin 15^\circ & 0 \\ -\sin 15^\circ & \cos 15^\circ & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & \theta_{12}^d & \theta_{13}^d \\ -\theta_{12}^d - \theta_{13}^d\theta_{23}^d & 1 & \theta_{23}^d \\ -\theta_{13}^d + \theta_{12}^d\theta_{23}^d & -\theta_{23}^d - \theta_{12}^d\theta_{13}^d & 1 \end{pmatrix}. \quad (562)$$

The relevant mixing elements are given as

$$\begin{aligned} V_{us} &\approx \theta_{12}^d \cos 15^\circ + \sin 15^\circ, \\ V_{ub} &\approx \theta_{13}^d \cos 15^\circ + \theta_{23}^d \sin 15^\circ, \\ V_{cb} &\approx -r_t\theta_{13}^d e^{i\rho} \sin 15^\circ + r_t\theta_{23}^d e^{i\rho} \cos 15^\circ - r_c\ , \\ V_{td} &\approx -r_c \sin 15^\circ e^{i\rho} - r_c(\theta_{12}^d + \theta_{13}^d\theta_{23}^d)e^{i\rho} \cos 15^\circ + r_t(-\theta_{13}^d + \theta_{12}^d\theta_{23}^d)\ . \end{aligned} \quad (563)$$

We can reproduce the experimental values with a parameter set

$$\rho = 123^\circ, \quad \theta_{12}^d = -0.0340, \quad \theta_{13}^d = 0.00626, \quad \theta_{23}^d = -0.00880, \quad (564)$$

by putting typical masses at the GUT scale $m_u = 1.04 \times 10^{-3}$GeV, $m_c = 302 \times 10^{-3}$ GeV, $m_t = 129$GeV [229].

In terms of a phase ρ, we can also estimate the magnitude of CP violation measure, Jarlskog invariant J_{CP} [230], which is given as

$$|J_{CP}| = |\mathrm{Im}\,\{V_{us}V_{cs}^* V_{ub}V_{cb}^*\}\,| \approx 3.06 \times 10^{-5}\ . \quad (565)$$

Our prediction is consistent with the experimental values $J_{CP} = 3.05^{+0.19}_{-0.20}$. We also show CP angles in the unitarity triangle, ϕ_1(or β), ϕ_2(or α) and ϕ_3(or γ),

$$\phi_1 = \arg\left(-\frac{V_{cd}V_{cb}^*}{V_{td}V_{tb}^*}\right), \qquad \phi_2 = \arg\left(-\frac{V_{td}V_{tb}^*}{V_{ud}V_{ub}^*}\right), \qquad \phi_3 = \arg\left(-\frac{V_{ud}V_{ub}^*}{V_{cd}V_{ccb}^*}\right). \quad (566)$$

Putting $\rho = 123^\circ$, we get $\sin 2\phi_1 = 0.693$, $\phi_2 = 89.4^\circ$ and $\phi_3 = 68.7^\circ$, which are compared with experimental values $\sin 2\phi_1 = 0.681 \pm 0.025$, $\phi_2 = (88^{+6}_{-5})^\circ$ and $\phi_3 = (77^{+30}_{-32})^\circ$.

15.4 $\Delta(54)$ flavor model

Certain classes of non-Abelian flavor symmetries can be derived from superstring theories. For example, D_4 and $\Delta(54)$ flavor symmetries can be obtained in heterotic orbifold models [14, 15, 16]. In addition to these flavor symmetries, the $\Delta(27)$ flavor symmetry can be derived from magnetized/intersecting D-brane models [17, 18, 19].

Here, we focus on the $\Delta(54)$ discrete symmetry. Although it includes several interesting aspects, few authors have considered up to now [70, 196, 197, 198]. The first aspect is that it consists of two types of Z_3 subgroups and an S_3 subgroup. The S_3 group is known as the minimal non-Abelian discrete symmetry, and the semi-direct product structure of $\Delta(54)$ between Z_3 and S_3 induces triplet irreducible representations. That suggests that the $\Delta(54)$ symmetry could lead to interesting models. The $\Delta(54)$ group has irreducible representations $\mathbf{1}_+$, $\mathbf{1}_-$, $\mathbf{2}_1$, $\mathbf{2}_2$, $\mathbf{2}_3$, $\mathbf{2}_4$, $\mathbf{3}_{1(1)}$, $\mathbf{3}_{1(2)}$, $\mathbf{3}_{2(1)}$, and $\mathbf{3}_{2(2)}$. There are four triplets and products of $\mathbf{3}_{1(1)} \times \mathbf{3}_{1(2)}$ and $\mathbf{3}_{2(1)} \times \mathbf{3}_{2(2)}$ lead to the trivial singlet.

	(l_e, l_μ, l_τ)	(e^c, μ^c, τ^c)	$(N_e^c, N_\mu^c, N_\tau^c)$	$h_{u(d)}$	χ_1	(χ_2, χ_3)	(χ_4, χ_5, χ_6)
$\Delta(54)$	$\mathbf{3}_{1(1)}$	$\mathbf{3}_{2(2)}$	$\mathbf{3}_{1(2)}$	$\mathbf{1}_+$	$\mathbf{1}_-$	$\mathbf{2}_1$	$\mathbf{3}_{1(2)}$

Table 69: Assignments of $\Delta(54)$ representations

15.4.1 Flavor model in lepton sector

Let us present the model of the lepton flavor with the $\Delta(54)$ group. The triplet representations of the group correspond to the three generations of leptons. The left-handed leptons (l_e, l_μ, l_τ), the right-handed charged leptons (e^c, μ^c, τ^c) and the right-handed neutrinos $(N_e^c, N_\mu^c, N_\tau^c)$ are assigned by $\mathbf{3}_{1(1)}$, $\mathbf{3}_{2(2)}$, and $\mathbf{3}_{1(2)}$, respectively. Since the product $\mathbf{3}_{1(1)} \times \mathbf{3}_{1(2)}$ includes the trivial singlet $\mathbf{1}_+$, only Dirac neutrino Yukawa couplings are allowed in tree level. On the other hand, charged leptons and the right-handed Majorana neutrinos cannot have mass terms unless new scalars χ_i are introduced in addition to the usual Higgs doublets, h_u and h_d. These new scalars are supposed to be $SU(2)$ gauge singlets. The gauge singlets χ_1, (χ_2, χ_3) and (χ_4, χ_5, χ_6) are assigned to $\mathbf{1}_-$, $\mathbf{2}_1$, and $\mathbf{3}_{1(2)}$ of the $\Delta(54)$ representations, respectively. The particle assignments of $\Delta(54)$ are summarized in Table 69. The usual Higgs doublets h_u and h_d are assigned to the trivial singlet $\mathbf{1}_+$ of $\Delta(54)$.

In this setup of the particle assignment, let us consider the superpotential of leptons at the leading order in terms of the cut-off scale Λ, which is taken to be the Planck scale. For charged leptons, the superpotential of the Yukawa sector respecting to $\Delta(54)$ symmetry is given as

$$\begin{aligned} w_l &= y_1^l(e^c l_e + \mu^c l_\mu + \tau^c l_\tau)\chi_1 h_d/\Lambda \\ &\quad + y_2^l \left[(\omega e^c l_e + \omega^2 \mu^c l_\mu + \tau^c l_\tau)\chi_2 - (e^c l_e + \omega^2 \mu^c l_\mu + \omega \tau^c l_\tau)\chi_3\right] h_d/\Lambda. \end{aligned} \tag{567}$$

For the right-handed Majorana neutrinos we can write the superpotential as follows:

$$\begin{aligned} w_N &= y_1(N_e^c N_e^c \chi_4 + N_\mu^c N_\mu^c \chi_5 + N_\tau^c N_\tau^c \chi_6) \\ &\quad + y_2 \left[(N_\mu^c N_\tau^c + N_\tau^c N_\mu^c)\chi_4 + (N_e^c N_\tau^c + N_\tau^c N_e^c)\chi_5 + (N_e^c N_\mu^c + N_\mu^c N_e^c)\chi_6\right]. \end{aligned} \quad (568)$$

The superpotential for the Dirac neutrinos has tree level contributions as

$$w_D = y_D \, (N_e^c l_e + N_\mu^c l_\mu + N_\tau^c l_\tau) h_u \, . \quad (569)$$

We assume that the scalar fields, $h_{u,d}$ and χ_i, develop their VEVs as follows:

$$\langle h_u \rangle = v_u, \ \langle h_d \rangle = v_d, \ \langle \chi_1 \rangle = u_1, \ \langle (\chi_2, \chi_3) \rangle = (u_2, u_3), \ \langle (\chi_4, \chi_5, \chi_6) \rangle = (u_4, u_5, u_6). \quad (570)$$

Then, we obtain the diagonal mass matrix for charged leptons

$$M_l = y_1^l v_d \begin{pmatrix} \alpha_1 & 0 & 0 \\ 0 & \alpha_1 & 0 \\ 0 & 0 & \alpha_1 \end{pmatrix} + y_2^l v_d \begin{pmatrix} \omega\alpha_2 - \alpha_3 & 0 & 0 \\ 0 & \omega^2\alpha_2 - \omega^2\alpha_3 & 0 \\ 0 & 0 & \alpha_2 - \omega\alpha_3 \end{pmatrix}, \quad (571)$$

while the right-handed Majorana mass matrix is given as

$$M_N = y_1 \Lambda \begin{pmatrix} \alpha_4 & 0 & 0 \\ 0 & \alpha_5 & 0 \\ 0 & 0 & \alpha_6 \end{pmatrix} + y_2 \Lambda \begin{pmatrix} 0 & \alpha_6 & \alpha_5 \\ \alpha_6 & 0 & \alpha_4 \\ \alpha_5 & \alpha_4 & 0 \end{pmatrix}, \quad (572)$$

and the Dirac mass matrix of neutrinos is obtianed as

$$M_D = y_D v_u \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad (573)$$

where we denote $\alpha_i = u_i/\Lambda$ $(i = 1-6)$. By using the seesaw mechanism $M_\nu = M_D^T M_N^{-1} M_D$, the neutrino mass matrix can be written as

$$\begin{aligned} M_\nu &= \frac{y_D^2 v_u^2}{\Lambda d} \begin{pmatrix} y_1^2\alpha_5\alpha_6 - y_2^2\alpha_4^2 & -y_1 y_2 \alpha_6^2 + y_2^2\alpha_4\alpha_5 & -y_1 y_2\alpha_5^2 + y_2^2\alpha_4\alpha_6 \\ -y_1 y_2\alpha_6^2 + y_2^2\alpha_4\alpha_5 & y_1^2\alpha_4\alpha_6 - y_2^2\alpha_5^2 & -y_1 y_2 \alpha_4^2 + y_2^2\alpha_5\alpha_6 \\ -y_1 y_2 \alpha_5^2 + y_2^2\alpha_4\alpha_6 & -y_1 y_2\alpha_4^2 + y_2^2\alpha_5\alpha_6 & y_1^2\alpha_4\alpha_5 - y_2^2\alpha_6^2 \end{pmatrix}, \\ d &= y_1^3\alpha_4\alpha_5\alpha_6 - y_1 y_2^2\alpha_4^3 - y_1 y_2^2 \alpha_5^3 - y_1 y_2^2\alpha_6^3 + 2 y_2^3 \alpha_4\alpha_5\alpha_6. \end{aligned} \quad (574)$$

Since the charged lepton mass matrix is diagonal one, we can simply get the mass eigenvalues as

$$\begin{pmatrix} m_e \\ m_\mu \\ m_\tau \end{pmatrix} = v_d \begin{pmatrix} 1 & \omega & -1 \\ 1 & \omega^2 & -\omega^2 \\ 1 & 1 & -\omega \end{pmatrix} \begin{pmatrix} y_1^\ell \alpha_1 \\ y_2^\ell \alpha_2 \\ y_2^\ell \alpha_3 \end{pmatrix}. \quad (575)$$

In order to estimate magnitudes of α_1, α_2 and α_3, we rewrite as

$$\begin{pmatrix} y_1^\ell \alpha_1 \\ y_2^\ell \alpha_2 \\ y_2^\ell \alpha_3 \end{pmatrix} = \frac{1}{3v_d} \begin{pmatrix} 1 & 1 & 1 \\ -\omega-1 & \omega & 1 \\ -1 & -\omega & \omega+1 \end{pmatrix} \begin{pmatrix} m_e \\ m_\mu \\ m_\tau \end{pmatrix}, \tag{576}$$

which gives the relation of $|y_2^\ell \alpha_2| = |y_2^\ell \alpha_3|$. Inserting the experimental values of the charged lepton masses and $v_d \simeq 55$GeV, which is given by taking $\tan\beta = 3$, we obtain numerical results

$$\begin{pmatrix} y_1^\ell \alpha_1 \\ y_2^\ell \alpha_2 \\ y_2^\ell \alpha_3 \end{pmatrix} = \begin{pmatrix} 1.14 \times 10^{-2} \\ 1.05 \times 10^{-2} e^{0.016 i\pi} \\ 1.05 \times 10^{-2} e^{0.32 i\pi} \end{pmatrix}. \tag{577}$$

Thus, it is found that $\alpha_i (i = 1, 2, 3)$ are of $\mathcal{O}(10^{-2})$ if the Yukawa couplings are order one.

In our model, the lepton mixing comes from the structure of the neutrino mass matrix of Eq.(574). In order to reproduce the maximal mixing between ν_μ and ν_τ, we take $\alpha_5 = \alpha_6$, and then we have

$$M_\nu = \frac{y_D^2 v_u^2}{\Lambda d} \begin{pmatrix} y_1^2\alpha_5^2 - y_2^2\alpha_4^2 & -y_1y_2\alpha_5^2 + y_2^2\alpha_4\alpha_5 & -y_1y_2\alpha_5^2 + y_2^2\alpha_4\alpha_5 \\ -y_1y_2\alpha_5^2 + y_2^2\alpha_4\alpha_5 & y_1^2\alpha_4\alpha_5 - y_2^2\alpha_5^2 & -y_1y_2\alpha_4^2 + y_2^2\alpha_5^2 \\ -y_1y_2\alpha_5^2 + y_2^2\alpha_4\alpha_5 & -y_1y_2\alpha_4^2 + y_2^2\alpha_5^2 & y_1^2\alpha_4\alpha_5 - y_2^2\alpha_5^2 \end{pmatrix}. \tag{578}$$

The tri-bimaximal mixing is realized by the condition of $M_\nu(1,1) + M_\nu(1,2) = M_\nu(2,2) + M_\nu(2,3)$ in Eq. (578), which turns to

$$(y_1 - y_2)(\alpha_4 - \alpha_5)(y_1\alpha_5 - y_2\alpha_4) = 0. \tag{579}$$

Therefore, we have three cases realizing the tri-bimaximal mixing in Eq.(578) as

$$y_1 = y_2, \qquad \alpha_4 = \alpha_5, \qquad y_1\alpha_5 = y_2\alpha_4. \tag{580}$$

Let us investigate the neutrino mass spectrum in these cases. In general the neutrino mass matrix with the tri-bimaximal mixing is expressed as the one in Eq.(493). Actually, the neutrino mass matrix of Eq.(578) is decomposed under the condition in Eq.(580) as follows. In the case of $\alpha_4 = \alpha_5$, the neutrino mass matrix is expressed as

$$M_\nu = \frac{y_D^2 v_u^2 \alpha_4^2 (y_1 - y_2)}{\Lambda d} \left[(y_1 + 2y_2) \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} - y_2 \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} \right]. \tag{581}$$

Therefore, it is found that neutrino masses are given as

$$\begin{aligned} \frac{m_1 + m_3}{2} &= \frac{y_D^2 v_u^2 \alpha_4^2 (y_1 - y_2)}{\Lambda d}(y_1 + 2y_2), \\ \frac{m_2 - m_1}{3} &= -\frac{y_D^2 v_u^2 \alpha_4^2}{\Lambda d}(y_1 - y_2) y_2, \\ m_1 - m_3 &= 0. \end{aligned} \tag{582}$$

In the case of $y_1 = y_2$, the mass matrix is decomposed as

$$M_\nu = \frac{y_D^2 y_1^2 v^2(\alpha_4-\alpha_5)}{\Lambda d}\left[\alpha_5\begin{pmatrix}1&1&1\\1&1&1\\1&1&1\end{pmatrix}-(\alpha_4+2\alpha_5)\begin{pmatrix}1&0&0\\0&0&1\\0&1&0\end{pmatrix}\right], \tag{583}$$

and we have

$$\begin{aligned} m_1+m_3 &= 0,\\ \frac{m_2-m_1}{3} &= \frac{y_D^2 y_1^2 v_u^2(\alpha_4-\alpha_5)}{\Lambda d}\alpha_5,\\ \frac{m_1-m_3}{2} &= -\frac{y_D^2 y_1^2 v_u^2(\alpha_4-\alpha_5)}{\Lambda d}(\alpha_4+2\alpha_5). \end{aligned} \tag{584}$$

In the last case of $y_1\alpha_5 = y_2\alpha_4$, we have

$$M_\nu = \frac{y_D^2 v_u^2}{\Lambda d}y_1^2\alpha_4\alpha_5\left(1-\frac{\alpha_5^3}{\alpha_4^3}\right)\left[\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}-\begin{pmatrix}1&0&0\\0&0&1\\0&1&0\end{pmatrix}\right]. \tag{585}$$

Then, we obtain

$$\begin{aligned} m_3 &= \frac{2y_D^2 v_u^2}{\Lambda d}y_1^2\alpha_4\alpha_5\left(1-\frac{\alpha_5^3}{\alpha_4^3}\right),\\ m_2 &= m_1 = 0\ . \end{aligned} \tag{586}$$

Thus, the tri-bimaximal mixing is not realized for arbitrary neutrino masses m_1, m_2 and m_3 in our model. In both conditions of $y_1 = y_2$ and $\alpha_4 = \alpha_5$, we have $|m_1| = |m_3|$, which leads to quasi-degenerate neutrino masses due to the condition of $\Delta m^2_{\rm atm} \gg \Delta m^2_{\rm sol}$. Therefore, we do not discuss these cases because we need fine-tuning of parameters in order to be consistent with the experimental data of the neutrino oscillations [3, 4, 5].

In the case of $y_1\alpha_5 = y_2\alpha_4$, the neutrino mass matrix turns to be

$$M_\nu = \frac{y_D^2 y_1^2 v_u^2}{\Lambda d}\begin{pmatrix}0&0&0\\0&\alpha_4\alpha_5-\alpha_5^4/\alpha_4^2&-\alpha_4\alpha_5+\alpha_5^4/\alpha_4^2\\0&-\alpha_4\alpha_5+\alpha_5^4/\alpha_4^2&\alpha_4\alpha_5-\alpha_5^4/\alpha_4^2\end{pmatrix}. \tag{587}$$

This neutrino matrix is a prototype which leads to the tri-bimaximal mixing with the mass hierarchy $m_3 \gg m_2 \geq m_1$, then we expect that realistic mass matrix is obtained near the condition $y_1\alpha_5 = y_2\alpha_4$.

Let us discuss the detail of the mass matrix (578). After rotating $\theta_{23} = 45^\circ$, we get

$$\frac{y_D^2 v_u^2}{\Lambda d}\begin{pmatrix} y_1^2\alpha_5^2-y_2^2\alpha_4^2 & \sqrt{2}(-y_1y_2\alpha_5^2+y_2^2\alpha_4\alpha_5) & 0\\ \sqrt{2}(-y_1y_2\alpha_5^2+y_2^2\alpha_4\alpha_5) & y_1^2\alpha_4\alpha_5-y_1y_2\alpha_4^2 & 0\\ 0 & 0 & y_1^2\alpha_4\alpha_5+y_1y_2\alpha_4^2-2y_2^2\alpha_5^2\end{pmatrix}, \tag{588}$$

which leads $\theta_{13} = 0$ and

$$\theta_{12} = \frac{1}{2}\arctan\frac{2\sqrt{2}y_2\alpha_5}{y_1\alpha_5 + y_2\alpha_4 - y_1\alpha_4} \qquad (y_2\alpha_4 \neq y_1\alpha_5). \tag{589}$$

Neutrino masses are given as

$$\begin{aligned} m_1 &= \frac{y_D^2 v_u^2}{\Lambda d}[y_1^2\alpha_5^2 - y_2^2\alpha_4^2 - \sqrt{2}(-y_1y_2\alpha_5^2 + y_2^2\alpha_4\alpha_5)\tan\theta_{12}], \\ m_2 &= \frac{y_D^2 v_u^2}{\Lambda d}[y_1^2\alpha_4\alpha_5 - y_1y_2\alpha_4^2 + \sqrt{2}(-y_1y_2\alpha_5^2 + y_2^2\alpha_4\alpha_5)\tan\theta_{12}], \\ m_3 &= \frac{y_D^2 v_u^2}{\Lambda d}[y_1^2\alpha_4\alpha_5 + y_1y_2\alpha_4^2 - 2y_2^2\alpha_5^2], \end{aligned} \tag{590}$$

which are reconciled with the normal hierarchy of neutrino masses in the case of $y_1\alpha_5 \simeq y_2\alpha_4$.

Let us estimate magnitudes of $\alpha_i (i = 4, 5, 6)$ by using Eq.(590). Suppose $\tilde{\alpha} = \alpha_4 \simeq \alpha_5 = \alpha_6$. If we take all Yukawa couplings to be order one, Eq.(590) turns to be $v_u^2 = \Lambda\tilde{\alpha}m_3$ because of $d \sim \tilde{\alpha}^3$. Putting $v_u \simeq 165$GeV $(\tan\beta = 3)$, $m_3 \simeq \sqrt{\Delta m^2_{\rm atm}} \simeq 0.05$eV, and $\Lambda = 2.43 \times 10^{18}$GeV, we obtain $\tilde{\alpha} = \mathcal{O}(10^{-4} - 10^{-3})$. Thus, values of $\alpha_i (i = 4, 5, 6)$ are enough suppressed to discuss perturbative series of higher mass operators.

We show our numerical analysis of neutrino masses and mixing angles in the normal mass hierarchy. Neglecting higher order corrections of mass matrices, we obtain the allowed region of parameters and predictions of neutrino masses and mixing angles. Here, we neglect the renomarization effect of the neutrino mass matrix because we suppose the normal hierarchy of neutrino masses and take $\tan\beta = 3$.

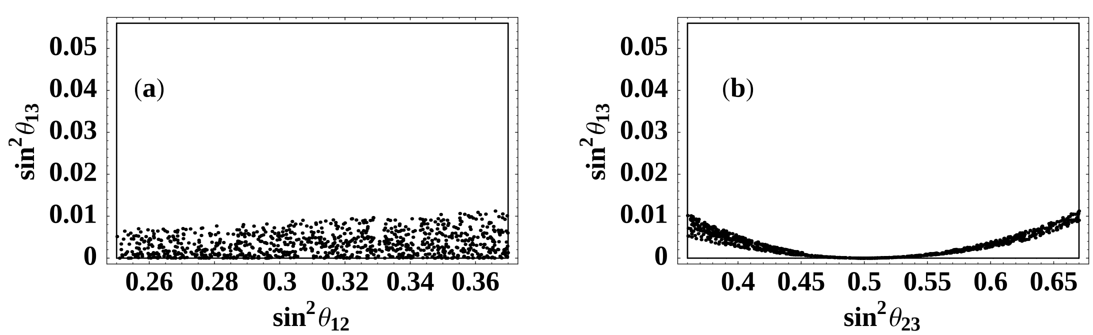

Figure 7: Prediction of the upper bound of $\sin^2\theta_{13}$ on (a) $\sin^2\theta_{12}$–$\sin^2\theta_{13}$ and (b) $\sin^2\theta_{23}$–$\sin^2\theta_{13}$ planes.

Input data of masses and mixing angles are taken in the region of 3σ of the experimental data [3, 4, 5] in Table 66 and $\Lambda = 2.43 \times 10^{18}$GeV is taken. We fix $y_D = y_1 = 1$ as a convention, and vary y_2/y_1. The change of y_D and y_1 is absorbed into the change of $\alpha_i (i = 4, 5, 6)$. If we take a smaller value of y_1, values of α_i scale up. On the other hand, if we take a smaller value of y_D, the magnitude of α_i scale down.

We can predict the deviation from the tri-bimaximal mixing. The remarkable prediction is given in the magnitude of $\sin^2\theta_{13}$. In Figures 7 (a) and (b), we plot the allowed region of mixing angles in planes of $\sin^2\theta_{12}$–$\sin^2\theta_{13}$ and $\sin^2\theta_{23}$–$\sin^2\theta_{13}$, respectively. It is found that the upper bound of $\sin^2\theta_{13}$ is 0.01. It is also found the strong correlation between $\sin^2\theta_{23}$ and $\sin^2\theta_{13}$. Unless θ_{23} is deviated from the maximal mixing considerably, θ_{13} remains to be tiny. Thus, the model reproduces the almost tri-bimaximal mixing in the parameter region around two vanishing neutrino masses. Therefore, the model is testable in the future neutrino experiments.

15.4.2 Comments on the $\Delta(54)$ flavor model

The $\Delta(54)$ symmetry can appear in heterotic string models on factorizable orbifolds including the T^2/Z_3 orbifold [15]. In these string models only singlets and triplets appear as fundamental modes, but doublets do not appear as fundamental modes. The doublet plays an role in our model, and such doublet could appear, e.g. as composite modes of triplets. On the other hand, doublets could appear as fundamental modes within the framework of magnetized/intersecting D-brane models.

As discussed in Eqs.(581)-(586), the tri-bimaximal mixing is not realized for arbitrary neutrino masses in our model. Parameters are adapted to get neutrino masses consistent with observed values of Δm^2_{atm} and Δm^2_{sol}. Then, the deviation from the tri-bimaximal mixing is predicted.

It is also useful to give the following comment on the $\Delta(27)$ flavor symmetry. Our mass matrix gives the same result in the $\Delta(27)$ flavor symmetry [188] where the type II seesaw is used.

We can present an alternative $\Delta(54)$ flavor model [198], in which the tri-bimaximal mixing is reproduced for arbitrary neutrino masses m_1, m_2 and m_3. The left-handed leptons (l_e, l_μ, l_τ), the right-handed charged leptons (e^c, μ^c, τ^c) are assigned to be $\mathbf{3}_{1(1)}$ and $\mathbf{3}_{2(2)}$, respectively. On the other hand, for right-handed neutrinos, N_e^c is assigned to be $\mathbf{1}_+$ and (N_μ^c, N_τ^c) are assigned to be $\mathbf{2}_2$. We introduce new scalars, which are supposed to be $SU(2)_L$ gauge singlets with vanishing $U(1)_Y$ charge. Gauge singlets χ_1, $\chi_1'(\chi_2, \chi_3)$, (χ_4, χ_5), (χ_6, χ_7, χ_8), and $(\chi_6', \chi_7', \chi_8')$ are assigned to be $\mathbf{1}_-$, $\mathbf{1}_1$, $\mathbf{2}_1$, $\mathbf{2}_2$, $\mathbf{3}_{1(2)}$, and $\mathbf{3}_{1(2)}$, respectively. We also introduce Z_3 symmetry and the non-trivial charge is assigned. The particle assignments of $\Delta(54)$ and Z_3 are summarized in Table 70.

	(l_e, l_μ, l_τ)	(e^c, μ^c, τ^c)	N_e	(N_μ, N_τ)	$h_{u(d)}$
$\Delta(54)$	$\mathbf{3}_{1(1)}$	$\mathbf{3}_{2(2)}$	$\mathbf{1}_+$	$\mathbf{2}_2$	$\mathbf{1}_+$
Z_3	1	ω	ω	1	1

	χ_1	χ_1'	(χ_2, χ_3)	(χ_4, χ_5)	(χ_6, χ_7, χ_8)	$(\chi_6', \chi_7', \chi_8')$
$\Delta(54)$	$\mathbf{1}_-$	$\mathbf{1}_+$	$\mathbf{2}_1$	$\mathbf{2}_2$	$\mathbf{3}_{1(2)}$	$\mathbf{3}_{1(2)}$
Z_3	ω^2	ω	ω^2	1	ω^2	1

Table 70: Assignments of $\Delta(54)$ and Z_3 representations, where ω is $e^{2\pi i/3}$.

In this particle assignment, the charged lepton mass matrix is diagonal while the right-handed Majorana mass matrix is given as

$$M_N = \begin{pmatrix} y_1^N \alpha_1' \Lambda & 0 & 0 \\ 0 & y_2^N \alpha_4 \Lambda & M \\ 0 & M & y_2^N \alpha_5 \Lambda \end{pmatrix}, \tag{591}$$

and the Dirac mass matrix of neutrinos is obtained as

$$M_D = y_1^D v_u \begin{pmatrix} \alpha_6 & \alpha_7 & \alpha_8 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} + y_2^D v_u \begin{pmatrix} 0 & 0 & 0 \\ \omega\alpha_8' & \omega^2\alpha_6' & \alpha_7' \\ \alpha_6 & \omega^2\alpha_7 & \omega\alpha_8 \end{pmatrix}, \tag{592}$$

where we denote $\alpha_i = u_i/\Lambda$. The tri-bimaximal mixing is realized by taking following alignments,

$$\alpha_5 = \omega\alpha_4 \ , \qquad \alpha_6 = \alpha_7 = \alpha_8. \tag{593}$$

By using the seesaw mechanism $M_\nu = M_D^T M_N^{-1} M_D$, the neutrino mass matrix can be derived as follows:

$$\begin{aligned} M_\nu = &\begin{pmatrix} 1 & 1 & 1 \\ 1 & \omega & \omega^2 \\ 1 & \omega^2 & \omega \end{pmatrix} \begin{pmatrix} y_1^D & 0 & 0 \\ 0 & y_2^D\omega & 0 \\ 0 & 0 & y_2^D \end{pmatrix} \begin{pmatrix} \frac{1}{M_1} & 0 & 0 \\ 0 & \frac{y^N\omega\alpha_4\Lambda}{(y^N\alpha_4\Lambda)^2\omega - M_2^2} & \frac{-M_2}{(y^N\alpha_4\Lambda)^2\omega - M_2^2} \\ 0 & \frac{-M_2}{(y^N\alpha_4\Lambda)^2\omega - M_2^2} & \frac{y^N\alpha_4\Lambda}{(y^N\alpha_4\Lambda)^2\omega - M_2^2} \end{pmatrix} \\ &\times \begin{pmatrix} y_1^D & 0 & 0 \\ 0 & y_2^D\omega & 0 \\ 0 & 0 & y_2^D \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 \\ 1 & \omega & \omega^2 \\ 1 & \omega^2 & \omega \end{pmatrix} \alpha_6^2 v_u^2. \end{aligned} \tag{594}$$

It can be rewritten as

$$M_\nu = 3c \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} + (a-b-c) \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} + 3b \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \tag{595}$$

where

$$a = \frac{(y_1^D)^2}{M_1}\alpha_6^2 v_u^2, \quad b = \frac{y^N (y_2^D)^2 \alpha_4 \Lambda}{(y^N\alpha_4\Lambda)^2\omega - M_2^2}\alpha_6^2 v_u^2, \quad c = \frac{-(y_2^D)^2 \omega M2}{(y^N\alpha_4\Lambda)^2\omega - M_2^2}\alpha_6^2 v_u^2. \tag{596}$$

Therefore, our neutrino mass matrix M_ν gives the tri-bimaximal mixing matrix U_{tribi} with mass eigenvalues as follows:

$$m_1 = 3(b+c), \quad m_2 = 3a, \quad m_3 = 3(c-b) \ . \tag{597}$$

15.5 Comment on alternative flavor mixing

In all the above models, the tri-bimaximal mixing of the lepton flavor is reproduced. However, there are other flavor mixing models, which are based on the golden ratio or the tri-maximal for the lepton flavor mixing.

The golden ratio is supposed to appear in the solar mixing angle θ_{12}. One example is proposed as $\tan\theta_{12} = 1/\phi$ where $\phi = (1+\sqrt{5})/2 \simeq 1.62$ [231]. The rotational icosahedral group, which is isomorphic to A_5, the alternating group of five elements, provides a natural context of the golden ratio $\tan\theta_{12} = 1/\phi$ [155]. The model was constructed in a minimal model at tree level in which the solar angle is related to the golden ratio, the atmospheric angle is maximal, and the reactor angle vanishes to leading order. The approach provides a rich setting in order to investigate the flavor puzzle of the Standard Model. Another context of the golden ratio $\cos\theta_{12} = \phi/2$ [232] is also proposed. The dihedral group D_{10} derives this ratio [233].

One can also consider the trimaximal lepton mixing [192], defined by $|U_{\alpha 2}|^2 = 1/3$ for $\alpha = e, \mu$, and so the mixing matrix U is given by using an arbitrary angle θ and a phase ϕ as follows:

$$U_{\rm tri} = \begin{pmatrix} \frac{2}{\sqrt{6}} & \frac{1}{\sqrt{3}} & 0 \\ -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} \cos\theta & 0 & \sin\theta e^{-i\phi} \\ 0 & 1 & 0 \\ -\sin\theta e^{i\phi} & 0 & \cos\theta \end{pmatrix} . \qquad (598)$$

This corresponds to a two-parameter lepton flavor mixing matrix. We present a model for the lepton sector in which tri-maximal mixing is enforced by softly broken discrete symmetries; one version of the model is based on the group $\Delta(27)$. A salient feature of the model is that no vacuum alignment is required.

The bimaximal neutrino mixing [234] is also studied in the context of the quark-lepton complementarity of mixing angles [235] in the S_4 model [83, 84].

15.6 Comments on other applications

Supersymmetric extension is one of interesting candidates for the physics beyond the standard model. Even if the theory is supersymmetric at high energy, supersymmetry must break above the weak scale. The supersymmetry breaking induces soft supersymmetry breaking terms such as gaugino masses, sfermion masses and scalar trilinear couplings, i.e. the so-called A-terms. Flavor symmetries control not only quark/lepton mass matrixes but also squark/slepton masses and their A-terms. Suppose that flavor symmetries are exact. When three families have quantum numbers different from each other under flavor symmetries, squark/slepton mass-squared matrices are diagonal. Furthermore, when two (three) of three families correspond to doublets (triplets) of flavor symmetries, their diagonal squark/slepton masses are degenerate. That would become an interesting prediction of a certain class of flavor models, which could be tested if the supersymmetry breaking scale is reachable by collider experiments. Flavor symmetries have similar effects on A-terms. These results are very important to suppress flavor changing neutral currents, which are constrained strongly by experiments. However, the flavor symmetry must break to lead to realistic quark/lepton mass matrices. Such breaking effects deform the above predictions. How much results are changed depends on breaking patterns. If masses of superpartners are of $\mathcal{O}(100)$ GeV, some models may be ruled out e.g. by experiments on flavor changing neutral currents. See e.g. Refs.[93, 41, 16, 172, 117, 198].

What is the origin of non-Abelian flavor symmetries ? Some of them are symmetries of geometrical solids. Thus, its origin may be geometrical aspects of extra dimensions. For example, it is found that the two-dimensional orbifold T^2/Z_2 with proper values of moduli has discrete symmetries such as A_4 and S_4 [12, 13].

Superstring theory is a promising candidate for unified theory including gravity, and predicts extra six dimensions. Superstring theory on a certain type of six-dimensional compact space realizes a discrete flavor symmetry. Such a string theory leads to stringy selection rules for allowed couplings among matter fields in four-dimensional effective field theory. Such stringy selection rules and geometrical symmetries as well as broken (continuous) gauge symmetries result in discrete flavor symmetries in superstring theory. For example, discrete flavor symmetries in heterotic orbifold models are studied in Refs. [14, 15, 16], and D_4 and $\Delta(54)$ are realized. Magnetized/intersecting D-brane models also realize the same flavor symmetries and other types such as $\Delta(27)$ [17, 18, 19]. Different types of non-Abelian flavor symmetries may be derived in other string models. Thus, such a study is quite important.

Alternatively, discrete flavor symmetries may be originated from continuous (gauge) symmetries [20, 21, 22].

At any rate, the experimental data of quark/lepton masses and mixing angles have no symmetry. Thus, non-Abelian flavor symmetries must be broken. The breaking direction is important, because the forms of mass matrices are determined by along which direction the flavor symmetries break. We need a proper breaking direction to derive realistic values of quark/lepton masses and mixing angles.

One way to fix the breaking direction is to analyze the potential minima of scalar fields with non-trivial representations of flavor symmetries. The number of the potential minima may be finite and in one of them the realistic breaking would happen. That is rather the conventional approach.

Another scenario to fix the breaking direction could be realized in theories with extra dimensions. One can impose the boundary conditions of matter fermions [59] and/or flavon scalars [242, 243, 244] in bulk such that zero models for some components of irreducible multiplets are projected out, that is, the symmetry breaking. If a proper component of a flavon multiplet remains, that can realize a realistic breaking direction.

16 Discussions and Summary

We have reviewed pedagogically non-Abelian discrete groups, which play an important role in the particle physics. We have shown group-theoretical aspects for many concrete groups explicitly, such as representations and their tensor products. We have shown them explicitly for non-Abelian discrete groups, S_N, A_N, T', D_N, Q_N, $\Sigma(2N^2)$, $\Delta(3N^2)$, T_7, $\Sigma(3N^3)$, and $\Delta(6N^2)$. We have explained pedagogically how to derive conjugacy classes, characters, representations and tensor products for these groups (with a finite number).

The origin of non-Abelian flavor symmetries is considered in the geometrical aspects of extra dimensions such as the two-dimensional orbifold T^2/Z_2 with proper values of moduli. Superstring theory on a certain type of six-dimensional compact space also realizes a discrete flavor symmetry. On the other hand, discrete subgroups of $SU(3)$ would be also interesting from the viewpoint of phenomenological applications for the flavor physics. Most of them have been shown for subgroups including doublets or triplets as the largest dimensional irreducible representations.

Here we comment on discrete subgroups of $SU(3)$ with larger dimensional irreducible representations, i.e. $\Sigma(168)$ and $\Sigma(n\phi)$ with $n = 36, 72, 216, 360$, where ϕ is *a group homomorphism*. The $\Sigma(168)$ [23, 25, 153, 237, 28] has one hundred and sixty eight elements, and has six irreducible representations; one singlet, two triplets(or one complex triplet), one sextet, one septet, and one octet. The $\Sigma(36\phi)$ [23, 25, 28] has one hundred and eight elements, and has fourteen irreducible representations; four singlets, eight triplets, and two quartets. The $\Sigma(72\phi)$ [23, 28] has two hundreds and sixteen elements, and has sixteen irreducible representations; four singlets, one doublet, eight triplets, two sextets(or one complex sextet), and one octet. The $\Sigma(216\phi)$, which is known as *Hessian group* [23, 25, 28], has one thousand and eighty elements, and has sixteen irreducible representations; three singlets, three doublets, seven triplets, six sextets, three octets, and two nonets. Readers can find details in ref. [28].

From the viewpoint of model building for the flavor physics, breaking patterns of discrete groups and decompositions of multiplets are important. We have summarized these breaking patterns of the non-Abelian discrete groups in section 13.

Symmetries at the tree-level can be broken in general by quantum effects, i.e. anomalies. Anomalies of continuous symmetries, in particular gauge symmetries, have been studied well. Here we have reviewed about anomalies of non-Abelian discrete symmetries by using the path integral approach. Also we have shown the anomaly-free conditions explicitly for several concrete groups. Similarly, readers could compute anomalies for other non-Abelian discrete symmetries. Those anomalies of non-Abelian discrete flavor symmetries would be controlled by string dynamics, when such flavor symmetries are originated from superstring theory. Then, studies on such anomalies would be important to provide us with a hint for the question: why there appear three families of quarks and leptons, because those anomalies are relevant to the generation number and the flavor structure.

We hope that this review contributes on the progress for particle physics with non-Abelian discrete groups.

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A Useful theorems

In this appendix, we give simple proofs of useful theorems. (See also e.g. Refs. [24, 26, 28].)

- **Lagrange's theorem**

The order N_H of a subgroup of a finite group G is a divisor of the order N_G of G.

Proof)

If $H = G$, the claim is trivial, $N_H = H_G$. Thus, we consider $H \neq G$. Let a_1 be an element of G, but be not contained in H. Here, we denote all of elements in H by $\{e = h_0, h_1, \cdots, h_{N_H-1}\}$. Then, we consider the products of a_1 and elements of H,

$$a_1 H = \{a_1, a_1 h_1, \cdots, a_1 h_{N_h-1}\}. \tag{599}$$

All of $a_1 h_i$ are different from each other. None of $a_1 h_i$ are contained in H. If $a_1 h_i = h_j$, we could find $a_1 = h_j h_i^{-1}$, that is, a_1 would be an element in H. Thus, the set $a_1 H$ includes the N_H elements. Next, let a_2 be an element of G, but be contained in neither H nor $a_1 H$. If $a_2 h_i = a_1 h_j$, the element a_2 would be written as $a_2 = a_1 h_j h_i^{-1}$, that is, an element of $a_1 H$. Thus, when $a_2 \notin H$ and $a_2 \notin a_1 H$, the set $a_2 H$ yields N_H new elements. We repeat this process. Then, we can decompose

$$G = H + a_1 H + \cdots + a_{m-1} H. \tag{600}$$

That implies $N_G = m N_H$. ∎

- **Theorem**

For a finite group, every representation is equivalent to a unitary representation.

Proof)

Every group element a is represented by a matrix $D(a)$, which acts on the vector space. We denote the basis of the representation vector space by $\{\boldsymbol{e}_1, \cdots, \boldsymbol{e}_d\}$. We consider two vectors, $\boldsymbol{v}$ and $\boldsymbol{w}$,

$$\boldsymbol{v} = \sum_{i=1}^{d} v_i \boldsymbol{e}_i, \qquad \boldsymbol{w} = \sum_{i=1}^{d} w_i \boldsymbol{e}_i. \tag{601}$$

We define the scalar product between $\mathbf{v}$ and $\mathbf{w}$ as

$$(\boldsymbol{v}, \boldsymbol{w}) = \sum_{i=1}^{d} v_i^* w_i. \tag{602}$$

Here, we define another scalar product by

$$\langle \boldsymbol{v}, \boldsymbol{w} \rangle = \frac{1}{N_G} \sum_{a \in G} (D(a)\boldsymbol{v}, D(a)\boldsymbol{w}). \tag{603}$$

Then, we find

$$\begin{aligned}\langle D(b)\boldsymbol{v}, D(b)\boldsymbol{w}\rangle &= \frac{1}{N_G}\sum_{a\in G}(D(b)D(a)\boldsymbol{v}, D(b)D(a)\boldsymbol{w}) \\ &= \frac{1}{N_G}\sum_{a\in G}(D(ba)\boldsymbol{v}, D(ba)\boldsymbol{w}) \\ &= \frac{1}{N_G}\sum_{c\in G}(D(c)\boldsymbol{v}, D(c)\boldsymbol{w}) \\ &= \langle \boldsymbol{v}, \boldsymbol{w}\rangle. \end{aligned} \tag{604}$$

That implies that $D(b)$ is unitary with respect to the scalar product $\langle \boldsymbol{v}, \boldsymbol{w}\rangle$. The orthogonal bases $\{\boldsymbol{e}_i\}$ and $\{\boldsymbol{e}'_i\}$ for the two scalar products $(\boldsymbol{v}, \boldsymbol{w})$ and $\langle \boldsymbol{v}, \boldsymbol{w}\rangle$ can be related by the linear transformation T as $\boldsymbol{e}'_i = T\boldsymbol{e}_i$, i.e. $(\boldsymbol{v}, \boldsymbol{w}) = \langle T\boldsymbol{v}, T\boldsymbol{w}\rangle$. We define $D'(g) = T^{-1}D(g)T$. Then, it is found that

$$\begin{aligned}(T^{-1}D(a)T\boldsymbol{v}, T^{-1}D(a)T\boldsymbol{w}) &= \langle D(a)T\boldsymbol{v}, D(a)T\boldsymbol{w}\rangle \\ &= \langle T\boldsymbol{v}, T\boldsymbol{w}\rangle \\ &= (\boldsymbol{v}, \boldsymbol{w}). \end{aligned} \tag{605}$$

That is, the matrix $D'(g)$ is unitary and is equivalent to $D(g)$. ∎

• **Schur's lemma**

(I) Let $D_1(g)$ and $D_2(g)$ be irreducible representations of G, which are inequivalent to each other. If

$$AD_1(g) = D_2(g)A, \qquad \forall g \in G, \tag{606}$$

the matrix A should vanish, $A = 0$.

(II) If

$$D(g)A = AD(g), \qquad \forall g \in G, \tag{607}$$

the matrix A should be proportional to the identity matrix I, i.e. $A = \lambda I$.

Proof) (I) We denote the representation vector spaces for $D_1(g)$ and $D_2(g)$ by V and W, respectively. Let the map A be a map $A : V \to W$ such that it satisfies (606). We consider the kernel of A,

$$Ker(A) = \{\boldsymbol{v} \in V | A\boldsymbol{v} = 0\}. \tag{608}$$

Let $\boldsymbol{v} \in Ker(A)$. Then, we have

$$AD_1(g)\boldsymbol{v} = D_2(g)A\boldsymbol{v} = 0. \tag{609}$$

It is found that $D_1(g)Ker(A) \subset Ker(A)$, that is, $Ker(A)$ is invariant under $D_1(g)$. Because $D_1(g)$ is irreducible, that implies that

$$Ker(A) = \{0\}, \qquad \text{or} \qquad Ker(A) = V. \tag{610}$$

The later, $Ker(A) = V$, can not be realized unless $A = 0$. Next, we consider the image

$$Im(A) = \{A\boldsymbol{v} | \boldsymbol{v} \in V\}. \tag{611}$$

We find

$$D_2(g)A\boldsymbol{v} = AD_1(g)\boldsymbol{v} \in Im(A). \tag{612}$$

That is, $Im(A)$ is invariant under $D_2(g)$. Because $D_2(g)$ is irreducible, that implies that

$$Im(A) = \{0\}, \qquad \text{or} \qquad Im(A) = W. \tag{613}$$

The former, $Im(A) = \{0\}$, can not be realized unless $A = 0$. As a result, it is found that A should satisfy

$$A = 0, \qquad \text{or} \qquad AD_1(g)A^{-1} = D_2(g). \tag{614}$$

The later means that the representations, $D_1(g)$ and $D_2(g)$, are equivalent to each other. Therefore, A should vanish, $A = 0$, if $D_1(g)$ and $D_2(g)$ are not equivalent. ∎

Proof)(II) Now, we consider the case with $D(g) = D_1(g) = D_2(g)$ and $V = W$. Here, A is a linear operators on V. The finite dimensional matrix A has at least one eigenvalue, because the characteristic equation $det(A - \lambda I) = 0$ has at lease one root, where λ is an eigenvalue. Then, Eq. (607) leads to

$$D(g)\,(A - \lambda I) = (A - \lambda I)\,D(g), \qquad \forall g \in G. \tag{615}$$

Using the above proof of Schur's lemma (I) and $Ker(A-\lambda I) \neq \{0\}$, we find $Ker(A-\lambda I) = V$, that is, $A - \lambda I = 0$. ∎

• **Theorem**

Let $D_\alpha(g)$ and $D_\beta(g)$ be irreducible representations of a group G on the d_α and d_β dimensional vector spaces. Then, they satisfy the following orthogonality relation,

$$\sum_{a\in G} D_\alpha(a)_{i\ell} D_\beta(a^{-1})_{mj} = \frac{N_G}{d_\alpha}\delta_{\alpha\beta}\delta_{ij}\delta_{\ell m}. \tag{616}$$

Proof)

We define

$$A = \sum_{a\in G} D_\alpha(a) B D_\alpha(a^{-1}), \tag{617}$$

where B is a $(d_\alpha \times d_\alpha)$ arbitrary matrix. We find $D(b)A = AD(b)$, since

$$\begin{aligned} D_\alpha(b)A &= \sum_{a\in G} D_\alpha(b)D_\alpha(a)BD_\alpha(a^{-1}) \\ &= \sum_{a\in G} D_\alpha(ba)BD_\alpha((ba)^{-1})D_\alpha(b) \\ &= \sum_{c\in G} D_\alpha(c)BD_\alpha(c^{-1})D_\alpha(b). \end{aligned} \tag{618}$$

That is, by use of Schur's lemma (II) it is found that the matrix A should be proportional to the $(d_\alpha \times d_\alpha)$ identity matrix. We choose $B_{ij} = \delta_{i\ell}\delta_{jm}$. Then, we obtain

$$A_{ij} = \sum_{a\in G} D_\alpha(a)_{i\ell} D_\alpha(a^{-1})_{mj}, \tag{619}$$

and right hand side (RHS) should be written by $\lambda(\ell, m)\delta_{ij}$, that is,

$$\sum_{a\in G} D_\alpha(a)_{i\ell} D_\alpha(a^{-1})_{mj} = \lambda(\ell, m)\delta_{ij}. \tag{620}$$

Furthermore, we compute the trace of both sides. The trace of RHS is computed as

$$\lambda(\ell, m)\mathrm{tr}\delta_{ij} = d_\alpha \lambda(\ell, m), \tag{621}$$

while the trace of left hand side (LHS) is obtained as

$$\begin{aligned} \sum_{i=1}^{d} \sum_{a\in G} D_\alpha(a)_{i\ell} D_\alpha(a^{-1})_{mi} &= \sum_{a\in G} D_\alpha(aa^{-1})_{\ell m} \\ &= N_G \delta_{\ell m}. \end{aligned} \tag{622}$$

By comparing these results, we obtain $\lambda(\ell, m) = \frac{N_G}{d_\alpha}\delta_{\ell m}$. Then, we find

$$\sum_{a\in G} D_\alpha(a)_{i\ell} D_\alpha(a^{-1})_{mj} = \frac{N_G}{d_\alpha}\delta_{ij}\delta_{\ell m}. \tag{623}$$

Similarly, we define

$$A^{(\alpha\beta)} = \sum_{a\in G} D_\alpha(a)BD_\beta(a^{-1}), \tag{624}$$

where $D_\alpha(a)$ and $D_\beta(a)$ are inequivalent to each other. Then, we find $D_\alpha(a)A = AD_\beta(a)$. Similarly to the previous analysis, using Schur's lemma (I), we can obtain

$$\sum_{a\in G} D_\alpha(a)_{i\ell} D_\beta(a^{-1})_{mj} = 0. \tag{625}$$

Thus, we can obtain Eq. (616). Furthermore, if the representation is unitary, Eq. (616) is written as

$$\sum_{a \in G} D_\alpha(a)_{i\ell} D^*_\beta(a)_{jm} = \frac{N_G}{d_\alpha} \delta_{\alpha\beta} \delta_{ij} \delta_{\ell m}. \qquad \blacksquare \tag{626}$$

Because of this orthogonality, we can expand an arbitrary function of a, $F(a)$, in terms of the matrix elements of irreducible representations

$$F(a) = \sum_{\alpha,j,k} c^\alpha_{j,k} D_\alpha(a)_{jk}. \tag{627}$$

• **Theorem**

The characters for $D_\alpha(g)$ and $D_\beta(g)$ representations, $\chi_\alpha(g)$ and $\chi_\beta(g)$, satisfy the following orthogonality relation,

$$\sum_{g \in G} \chi_{D_\alpha}(g)^* \chi_{D_\beta}(g) = N_G \delta_{\alpha\beta}. \tag{628}$$

Proof)

From Eq. (626) we obtain

$$\sum_{g \in G} D_\alpha(g)_{ii} D^*_\beta(g)_{jj} = \frac{N_G}{d_\alpha} \delta_{\alpha\beta} \delta_{ij}. \tag{629}$$

Thus, by summing over all i and j, we obtain Eq. (628). $\blacksquare$

The **class function** is defined as a function of a, $F(a)$, which satisfies

$$F(g^{-1}ag) = F(a), \qquad \forall g \in G. \tag{630}$$

• **Theorem**

The number of irreducible representations is equal to the number of conjugacy classes.

Proof) The class function can also be expanded in terms of the matrix elements of the irreducible representations as (627). Then, it is found that

$$\begin{aligned} F(a) &= \frac{1}{N_G} \sum_{g \in G} F(g^{-1}ag) \\ &= \frac{1}{N_G} \sum_{g \in G} \sum_{\alpha,j,k} c^\alpha_{j,k} D_\alpha(g^{-1}ag)_{jk} \\ &= \frac{1}{N_G} \sum_{g \in G} \sum_{\alpha,j,k} c^\alpha_{j,k} \left(D_\alpha(g^{-1}) D_\alpha(a) D_\alpha(g) \right)_{jk}. \end{aligned} \tag{631}$$

By using the orthogonality relation (626), we obtain

$$
\begin{aligned}
F(a) &= \sum_{\alpha,j,\ell} \frac{1}{d_\alpha} c^{\alpha}_{j,j} D_\alpha(a)_{\ell\ell} \\
&= \sum_{\alpha,j} \frac{1}{d_\alpha} c^{\alpha}_{j,j} \chi_\alpha(a). \qquad (632)
\end{aligned}
$$

That is, any class function, $F(a)$, which is constant on conjugacy classes, can be expanded by the characters $\chi_\alpha(a)$. That implies that the number of irreducible representations is equal to the number of conjugacy classes. ∎

• **Theorem**
The characters satisfy the following orthogonality relation,

$$
\sum_\alpha \chi_{D_\alpha}(C_i)^* \chi_{D_\alpha}(C_j) = \frac{N_G}{n_i} \delta_{C_i C_j}, \qquad (633)
$$

where C_i and C_j denote the conjugacy classes and n_i is the number of elements in the conjugacy class C_i.

Proof)
We define the following matrix $V_{i\alpha}$,

$$
V_{i\alpha} = \sqrt{\frac{n_i}{N_G}} \chi_\alpha(C_i), \qquad (634)
$$

where n_i is the number of elements in the conjugacy class C_i. Note that i and α label the conjugacy class C_i and the irreducible representation, respectively. The matrix $V_{i\alpha}$ is a square matrix because the number of irreducible representations is equal to the number of conjugacy classes. By use of $V_{i\alpha}$, the orthogonality relation (628) can be rewritten as $V^\dagger V = 1$, that is, V is unitary. Thus, we also obtain $VV^\dagger = 1$. That means Eq. (633). ∎

B Representations of S_4 in several bases

For the S_4 group, several bases of representations have been used in the literature. Most of group-theoretical aspects such as conjugacy classes and characters are independent of the basis of representations. Tensor products are also independent of the basis. For example, we always have

$$\mathbf{2} \otimes \mathbf{2} = \mathbf{1}_1 \oplus \mathbf{1}_2 \oplus \mathbf{2}, \tag{635}$$

in any basis. However, it depends on the basis of representation how this equation is written by components. For example, the singlets $\mathbf{1}_1$ and $\mathbf{1}_2$ in RHS are represented by components of $\mathbf{2}$ in LHS, but their forms depend on the basis of representations as we will see below. For applications, it is useful to show explicitly the transformation of bases and tensor products for several bases. That is shown below.

First, we show the basis in section 3.2. All of the S_4 elements are written by products of the generators b_1 and d_4, which satisfy

$$(b_1)^3 = (d_4)^4 = e, \quad d_4(b_1)^2 d_4 = b_1, \quad d_4 b_1 d_4 = b_1 (d_4)^2 b_1 \ . \tag{636}$$

These generators are represented on $\mathbf{2}$, $\mathbf{3}$ and $\mathbf{3}'$ as follows,

$$b_1 = \begin{pmatrix} \omega & 0 \\ 0 & \omega^2 \end{pmatrix}, \quad d_4 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \text{on } \mathbf{2}, \tag{637}$$

$$b_1 = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \quad d_4 = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}, \qquad \text{on } \mathbf{3}, \tag{638}$$

$$b_1 = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \quad d_4 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}, \qquad \text{on } \mathbf{3}'. \tag{639}$$

Next, we consider another basis, which is used e.g. in Ref. [77]. Following Ref. [77], we denote the generators b_1 and d_4 by $b = b_1$ and $a = d_4$. In this basis, the generators, a and b, are represented as

$$a = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \quad b = -\frac{1}{2}\begin{pmatrix} 1 & \sqrt{3} \\ -\sqrt{3} & 1 \end{pmatrix}, \qquad \text{on } \mathbf{2}, \tag{640}$$

$$a = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}, \quad b = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \qquad \text{on } \mathbf{3}_1, \tag{641}$$

$$a = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}, \quad b = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \qquad \text{on } \mathbf{3}_2, \tag{642}$$

where we define as $\mathbf{3}_1 \equiv \mathbf{3}$ and $\mathbf{3}_2 \equiv \mathbf{3}'$ hereafter. These generators, a and b, are represented in the real basis. On the other hand, the above generators, b_1 and d_4, are represented in the complex basis. These bases for $\mathbf{2}$ are transformed by the unitary transformation, $U^\dagger g U$, where

$$U = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & i \\ -1 & i \end{pmatrix}. \tag{643}$$

That is, the elements a and b are written by b_1 and d_4 as

$$b = U^\dagger b_1 U = -\frac{1}{2} \begin{pmatrix} 1 & \sqrt{3} \\ -\sqrt{3} & 1 \end{pmatrix}, \quad a = U^\dagger d_4 U = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \tag{644}$$

in the real basis. For the triplets, the (b_1, d_4) basis is the same as the (b, a) basis.

Therefore, the multiplication rules are obtained as follows:

$$\begin{pmatrix} a_1 \\ a_2 \end{pmatrix}_{\mathbf{2}} \otimes \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}_{\mathbf{2}} = (a_1 b_1 + a_2 b_2)_{\mathbf{1}_1} \oplus (-a_1 b_2 + a_2 b_1)_{\mathbf{1}_2} \oplus \begin{pmatrix} a_1 b_2 + a_2 b_1 \\ a_1 b_1 - a_2 b_2 \end{pmatrix}_{\mathbf{2}}, \tag{645}$$

$$\begin{pmatrix} a_1 \\ a_2 \end{pmatrix}_{\mathbf{2}} \otimes \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}_{\mathbf{3}_1} = \begin{pmatrix} a_2 b_1 \\ -\frac{1}{2}(\sqrt{3} a_1 b_2 + a_2 b_2) \\ \frac{1}{2}(\sqrt{3} a_1 b_3 - a_2 b_3) \end{pmatrix}_{\mathbf{3}_1} \oplus \begin{pmatrix} a_1 b_1 \\ \frac{1}{2}(\sqrt{3} a_2 b_2 - a_1 b_2) \\ -\frac{1}{2}(\sqrt{3} a_2 b_3 + a_1 b_3) \end{pmatrix}_{\mathbf{3}_2}, \tag{646}$$

$$\begin{pmatrix} a_1 \\ a_2 \end{pmatrix}_{\mathbf{2}} \otimes \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}_{\mathbf{3}_2} = \begin{pmatrix} a_1 b_1 \\ \frac{1}{2}(\sqrt{3} a_2 b_2 - a_1 b_2) \\ -\frac{1}{2}(\sqrt{3} a_2 b_3 + a_1 b_3) \end{pmatrix}_{\mathbf{3}_1} \oplus \begin{pmatrix} a_2 b_1 \\ -\frac{1}{2}(\sqrt{3} a_1 b_2 + a_2 b_2) \\ \frac{1}{2}(\sqrt{3} a_1 b_3 - a_2 b_3) \end{pmatrix}_{\mathbf{3}_2}, \tag{647}$$

$$\begin{aligned} \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}_{\mathbf{3}_1} \otimes \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}_{\mathbf{3}_1} &= (a_1 b_1 + a_2 b_2 + a_3 b_3)_{\mathbf{1}_1} \oplus \begin{pmatrix} \frac{1}{\sqrt{2}}(a_2 b_2 - a_3 b_3) \\ \frac{1}{\sqrt{6}}(-2a_1 b_1 + a_2 b_2 + a_3 b_3) \end{pmatrix}_{\mathbf{2}} \\ &\oplus \begin{pmatrix} a_2 b_3 + a_3 b_2 \\ a_1 b_3 + a_3 b_1 \\ a_1 b_2 + a_2 b_1 \end{pmatrix}_{\mathbf{3}_1} \oplus \begin{pmatrix} a_3 b_2 - a_2 b_3 \\ a_1 b_3 - a_3 b_1 \\ a_2 b_1 - a_1 b_2 \end{pmatrix}_{\mathbf{3}_2}, \end{aligned} \tag{648}$$

$$\begin{aligned} \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}_{\mathbf{3}_2} \otimes \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}_{\mathbf{3}_2} &= (a_1 b_1 + a_2 b_2 + a_3 b_3)_{\mathbf{1}_1} \oplus \begin{pmatrix} \frac{1}{\sqrt{2}}(a_2 b_2 - a_3 b_3) \\ \frac{1}{\sqrt{6}}(-2a_1 b_1 + a_2 b_2 + a_3 b_3) \end{pmatrix}_{\mathbf{2}} \\ &\oplus \begin{pmatrix} a_2 b_3 + a_3 b_2 \\ a_1 b_3 + a_3 b_1 \\ a_1 b_2 + a_2 b_1 \end{pmatrix}_{\mathbf{3}_1} \oplus \begin{pmatrix} a_3 b_2 - a_2 b_3 \\ a_1 b_3 - a_3 b_1 \\ a_2 b_1 - a_1 b_2 \end{pmatrix}_{\mathbf{3}_2}, \end{aligned} \tag{649}$$

$$\begin{aligned} \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}_{\mathbf{3}_1} \otimes \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}_{\mathbf{3}_2} &= (a_1 b_1 + a_2 b_2 + a_3 b_3)_{\mathbf{1}_2} \oplus \begin{pmatrix} \frac{1}{\sqrt{6}}(2a_1 b_1 - a_2 b_2 - a_3 b_3) \\ \frac{1}{\sqrt{2}}(a_2 b_2 - a_3 b_3) \end{pmatrix}_{\mathbf{2}} \\ &\oplus \begin{pmatrix} a_3 b_2 - a_2 b_3 \\ a_1 b_3 - a_3 b_1 \\ a_2 b_1 - a_1 b_2 \end{pmatrix}_{\mathbf{3}_1} \oplus \begin{pmatrix} a_2 b_3 + a_3 b_2 \\ a_1 b_3 + a_3 b_1 \\ a_1 b_2 + a_2 b_1 \end{pmatrix}_{\mathbf{3}_2}. \end{aligned} \tag{650}$$

Next, we consider a different basis, which is used, e.g. in Ref. [73], with the generator s and t corresponding to d_4 and b_1, respectively. These generators are represented

$$s = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad t = \begin{pmatrix} \omega & 0 \\ 0 & \omega^2 \end{pmatrix}, \qquad \text{on } \mathbf{2}, \tag{651}$$

$$s = \frac{1}{3}\begin{pmatrix} -1 & 2\omega & 2\omega^2 \\ 2\omega & 2\omega^2 & -1 \\ 2\omega^2 & -1 & 2\omega \end{pmatrix}, \quad t = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \omega^2 & 0 \\ 0 & 0 & \omega \end{pmatrix}, \qquad \text{on } \mathbf{3}_1, \tag{652}$$

$$s = \frac{1}{3}\begin{pmatrix} 1 & -2\omega & -2\omega^2 \\ -2\omega & -2\omega^2 & 1 \\ -2\omega^2 & 1 & -2\omega \end{pmatrix}, \quad t = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \omega^2 & 0 \\ 0 & 0 & \omega \end{pmatrix}, \qquad \text{on } \mathbf{3}_2. \tag{653}$$

The doublet of this basis [73] is the same as the (d_4, b_1) basis. In the representations $\mathbf{3}_1$ and $\mathbf{3}_2$, the (s, t) basis and (d_4, b_1) basis are transformed by the following unitary matrix U_ω:

$$U_\omega = \frac{1}{\sqrt{3}}\begin{pmatrix} 1 & 1 & 1 \\ 1 & \omega & \omega^2 \\ 1 & \omega^2 & \omega \end{pmatrix}, \tag{654}$$

which is the so-called magic matrix. That is, the elements s and t are written by d_4 and b_1 as

$$s = U_\omega^\dagger d_4 U_\omega = \frac{1}{3}\begin{pmatrix} -1 & 2\omega & 2\omega^2 \\ 2\omega & 2\omega^2 & -1 \\ 2\omega^2 & -1 & 2\omega^2 \end{pmatrix}, \quad t = U_\omega^\dagger b_1 U_\omega = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \omega^2 & 0 \\ 0 & 0 & \omega \end{pmatrix}. \tag{655}$$

For $\mathbf{3}_2$, we also find s and t in the same way.

Therefore, the multiplication rules are obtained as follows:

$$\begin{pmatrix} a_1 \\ a_2 \end{pmatrix}_{\mathbf{2}} \otimes \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}_{\mathbf{2}} = (a_1 b_2 + a_2 b_1)_{\mathbf{1}_1} \oplus (a_1 b_2 - a_2 b_1)_{\mathbf{1}_2} \oplus \begin{pmatrix} a_2 b_2 \\ a_1 b_1 \end{pmatrix}_{\mathbf{2}}, \tag{656}$$

$$\begin{pmatrix} a_1 \\ a_2 \end{pmatrix}_{\mathbf{2}} \otimes \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}_{\mathbf{3}_1} = \begin{pmatrix} a_1 b_2 + a_2 b_3 \\ a_1 b_3 + a_2 b_1 \\ a_1 b_1 + a_2 b_2 \end{pmatrix}_{\mathbf{3}_1} \oplus \begin{pmatrix} a_1 b_2 - a_2 b_3 \\ a_1 b_3 - a_2 b_1 \\ a_1 b_1 - a_2 b_2 \end{pmatrix}_{\mathbf{3}_2}, \tag{657}$$

$$\begin{pmatrix} a_1 \\ a_2 \end{pmatrix}_{\mathbf{2}} \otimes \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}_{\mathbf{3}_2} = \begin{pmatrix} a_1 b_2 - a_2 b_3 \\ a_1 b_3 - a_2 b_1 \\ a_1 b_1 - a_2 b_2 \end{pmatrix}_{\mathbf{3}_1} \oplus \begin{pmatrix} a_1 b_2 + a_2 b_3 \\ a_1 b_3 + a_2 b_1 \\ a_1 b_1 + a_2 b_2 \end{pmatrix}_{\mathbf{3}_2}, \tag{658}$$

$$\begin{aligned} \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}_{\mathbf{3}_1} \otimes \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}_{\mathbf{3}_1} &= (a_1 b_1 + a_2 b_3 + a_3 b_2)_{\mathbf{1}_1} \oplus \begin{pmatrix} a_2 b_2 + a_1 b_3 + a_3 b_1 \\ a_3 b_3 + a_1 b_2 + a_2 b_1 \end{pmatrix}_{\mathbf{2}} \\ &\oplus \begin{pmatrix} 2a_1 b_1 - a_2 b_3 - a_3 b_2 \\ 2a_3 b_3 - a_1 b_2 - a_2 b_1 \\ 2a_2 b_2 - a_1 b_3 - a_3 b_1 \end{pmatrix}_{\mathbf{3}_1} \oplus \begin{pmatrix} a_2 b_3 - a_3 b_2 \\ a_1 b_2 - a_2 b_1 \\ a_3 b_1 - a_1 b_3 \end{pmatrix}_{\mathbf{3}_2}, \end{aligned} \tag{659}$$

$$\begin{aligned} \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}_{\mathbf{3}_2} \otimes \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}_{\mathbf{3}_2} &= (a_1 b_1 + a_2 b_3 + a_3 b_2)_{\mathbf{1}_1} \oplus \begin{pmatrix} a_2 b_2 + a_1 b_3 + a_3 b_1 \\ a_3 b_3 + a_1 b_2 + a_2 b_1 \end{pmatrix}_{\mathbf{2}} \\ &\oplus \begin{pmatrix} 2a_1 b_1 - a_2 b_3 - a_3 b_2 \\ 2a_3 b_3 - a_1 b_2 - a_2 b_1 \\ 2a_2 b_2 - a_1 b_3 - a_3 b_1 \end{pmatrix}_{\mathbf{3}_1} \oplus \begin{pmatrix} a_2 b_3 - a_3 b_2 \\ a_1 b_2 - a_2 b_1 \\ a_3 b_1 - a_1 b_3 \end{pmatrix}_{\mathbf{3}_2}, \end{aligned} \tag{660}$$

$$\begin{aligned} \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}_{\mathbf{3}_1} \otimes \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}_{\mathbf{3}_2} &= (a_1 b_1 + a_2 b_3 + a_3 b_2)_{\mathbf{1}_2} \oplus \begin{pmatrix} a_2 b_2 + a_1 b_3 + a_3 b_1 \\ -a_3 b_3 - a_1 b_2 - a_2 b_1 \end{pmatrix}_{\mathbf{2}} \\ &\oplus \begin{pmatrix} a_2 b_3 - a_3 b_2 \\ a_1 b_2 - a_2 b_1 \\ a_3 b_1 - a_1 b_3 \end{pmatrix}_{\mathbf{3}_1} \oplus \begin{pmatrix} 2a_1 b_1 - a_2 b_3 - a_3 b_2 \\ 2a_3 b_3 - a_1 b_2 - a_2 b_1 \\ 2a_2 b_2 - a_1 b_3 - a_3 b_1 \end{pmatrix}_{\mathbf{3}_2}. \end{aligned} \tag{661}$$

Here, we consider another basis, which is used, e.g. in Ref. [83], with the generator $\tilde{t}$ and $\tilde{s}$ satisfying

$$\tilde{t}^4 = \tilde{s}^2 = (\tilde{s}\tilde{t})^3 = (\tilde{t}\tilde{s})^3 = e. \tag{662}$$

These generators are represented as

$$\tilde{t} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \tilde{s} = \frac{1}{2}\begin{pmatrix} -1 & \sqrt{3} \\ \sqrt{3} & 1 \end{pmatrix}, \quad \tilde{s}\tilde{t} = \frac{1}{2}\begin{pmatrix} -1 & -\sqrt{3} \\ \sqrt{3} & -1 \end{pmatrix}, \qquad \text{on } \mathbf{2}, \tag{663}$$

$$\tilde{t} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -i & 0 \\ 0 & 0 & i \end{pmatrix}, \quad \tilde{s} = \begin{pmatrix} 0 & -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{\sqrt{2}} & -\frac{1}{2} & \frac{1}{2} \end{pmatrix},$$

$$\tilde{s}\tilde{t} = \begin{pmatrix} 0 & \frac{i}{\sqrt{2}} & -\frac{i}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{i}{2} & -\frac{i}{2} \\ \frac{1}{\sqrt{2}} & \frac{i}{2} & \frac{i}{2} \end{pmatrix}, \qquad \text{on } \mathbf{3}_1, \tag{664}$$

$$\tilde{t} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & i & 0 \\ 0 & 0 & -i \end{pmatrix}, \quad \tilde{s} = \begin{pmatrix} 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{2} & \frac{1}{2} \\ \frac{1}{\sqrt{2}} & \frac{1}{2} & -\frac{1}{2} \end{pmatrix},$$

$$\tilde{s}\tilde{t} = \begin{pmatrix} 0 & \frac{i}{\sqrt{2}} & -\frac{i}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{i}{2} & -\frac{i}{2} \\ \frac{1}{\sqrt{2}} & \frac{i}{2} & \frac{i}{2} \end{pmatrix}, \qquad \text{on } \mathbf{3}_2. \tag{665}$$

For the representation **2**, the following unitary transformation matrix U_{doublet}:

$$U_{\text{doublet}} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & i \\ 1 & -i \end{pmatrix}, \tag{666}$$

is used and the elements $\tilde{t}$ and $\tilde{s}\tilde{t}$ are written by d_1 and b_1 as

$$\tilde{t} = U^\dagger_{\text{doublet}} d_4 U_{\text{doublet}} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \tilde{s}\tilde{t} = U^\dagger_{\text{doublet}} b_1 U_{\text{doublet}} = \frac{1}{2} \begin{pmatrix} -1 & -\sqrt{3} \\ \sqrt{3} & -1 \end{pmatrix}. \tag{667}$$

On the other hand, for the representation $\mathbf{3}_1$ and $\mathbf{3}_2$, the following unitary transformation matrix U_{triplet}:

$$U_{\text{triplet}} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 0 & \frac{i}{\sqrt{2}} & -\frac{i}{\sqrt{2}} \end{pmatrix}, \tag{668}$$

is used. For $\mathbf{3}_1$, the elements $\tilde{t}$ and $\tilde{s}\tilde{t}$ are written by d_4 and b_1 as

$$\tilde{t} = U^\dagger_{\text{triplet}} d_4 U_{\text{triplet}} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -i & 0 \\ 0 & 0 & i \end{pmatrix},$$

$$\tilde{s}\tilde{t} = U^\dagger_{\text{triplet}} b_1 U_{\text{triplet}} = \begin{pmatrix} 0 & \frac{i}{\sqrt{2}} & -\frac{i}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{i}{2} & -\frac{i}{2} \\ \frac{1}{\sqrt{2}} & \frac{i}{2} & \frac{i}{2} \end{pmatrix}. \tag{669}$$

For $\mathbf{3}_2$, we also find the same transformations.

Therefore, the multiplication rules are as follows:

$$\begin{pmatrix} a_1 \\ a_2 \end{pmatrix}_{\mathbf{2}} \otimes \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}_{\mathbf{2}} = (a_1b_1 + a_2b_2)_{\mathbf{1}_1} \oplus (a_1b_2 - a_2b_1)_{\mathbf{1}_2} \oplus \begin{pmatrix} a_2b_2 - a_1b_1 \\ a_1b_2 + a_2b_1 \end{pmatrix}_{\mathbf{2}} , \tag{670}$$

$$\begin{pmatrix} a_1 \\ a_2 \end{pmatrix}_{\mathbf{2}} \otimes \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}_{\mathbf{3}_1} = \begin{pmatrix} a_1b_1 \\ \frac{\sqrt{3}}{2}a_2b_3 - \frac{1}{2}a_1b_2 \\ \frac{\sqrt{3}}{2}a_2b_2 - a_1b_3 \end{pmatrix}_{\mathbf{3}_1} \oplus \begin{pmatrix} -a_2b_1 \\ \frac{\sqrt{3}}{2}a_1b_3 + \frac{1}{2}a_2b_2 \\ \frac{\sqrt{3}}{2}a_1b_2 + \frac{1}{2}a_2b_3 \end{pmatrix}_{\mathbf{3}_2} , \tag{671}$$

$$\begin{pmatrix} a_1 \\ a_2 \end{pmatrix}_{\mathbf{2}} \otimes \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}_{\mathbf{3}_2} = \begin{pmatrix} -a_2b_1 \\ \frac{\sqrt{3}}{2}a_1b_3 + \frac{1}{2}a_2b_2 \\ \frac{\sqrt{3}}{2}a_1b_2 + \frac{1}{2}a_2b_3 \end{pmatrix}_{\mathbf{3}_1} \oplus \begin{pmatrix} a_1b_1 \\ \frac{\sqrt{3}}{2}a_2b_3 - \frac{1}{2}a_1b_2 \\ \frac{\sqrt{3}}{2}a_2b_2 - a_1b_3 \end{pmatrix}_{\mathbf{3}_2} , \tag{672}$$

$$\begin{aligned} \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}_{\mathbf{3}_1} \otimes \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}_{\mathbf{3}_1} &= (a_1b_1 + a_2b_3 + a_3b_2)_{1_1} \oplus \begin{pmatrix} a_1b_1 - \frac{1}{2}(a_2b_3 + a_3b_2) \\ \frac{\sqrt{3}}{2}(a_2b_2 + a_3b_3) \end{pmatrix}_{\mathbf{2}} \\ &\oplus \begin{pmatrix} a_3b_3 - a_2b_2 \\ a_1b_3 + a_3b_1 \\ -a_1b_2 - a_2b_1 \end{pmatrix}_{\mathbf{3}_1} \oplus \begin{pmatrix} a_3b_2 - a_2b_3 \\ a_2b_1 - a_1b_2 \\ a_1b_3 - a_3b_1 \end{pmatrix}_{\mathbf{3}_2} , \end{aligned} \tag{673}$$

$$\begin{aligned} \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}_{\mathbf{3}_2} \otimes \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}_{\mathbf{3}_2} &= (a_1b_1 + a_2b_3 + a_3b_2)_{\mathbf{1}_1} \oplus \begin{pmatrix} a_1b_1 - \frac{1}{2}(a_2b_3 + a_3b_2) \\ \frac{\sqrt{3}}{2}(a_2b_2 + a_3b_3) \end{pmatrix}_{\mathbf{2}} \\ &\oplus \begin{pmatrix} a_3b_3 - a_2b_2 \\ a_1b_3 + a_3b_1 \\ -a_1b_2 - a_2b_1 \end{pmatrix}_{\mathbf{3}_1} \oplus \begin{pmatrix} a_3b_2 - a_2b_3 \\ a_2b_1 - a_1b_2 \\ a_1b_3 - a_3b_1 \end{pmatrix}_{\mathbf{3}_2} , \end{aligned} \tag{674}$$

$$\begin{aligned} \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}_{\mathbf{3}_1} \otimes \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}_{\mathbf{3}_2} &= (a_1b_1 + a_2b_3 + a_3b_2)_{\mathbf{1}_2} \\ &\oplus \left(\frac{\sqrt{3}}{2}(a_2b_2 + a_3b_3) - a_1b_1 + \frac{1}{2}(a_2b_3 + a_3b_2) \right)_{\mathbf{2}} \\ &\oplus \begin{pmatrix} a_3b_2 - a_2b_3 \\ a_2b_1 - a_1b_2 \\ a_1b_3 - a_3b_1 \end{pmatrix}_{\mathbf{3}_1} \oplus \begin{pmatrix} a_3b_3 - a_2b_2 \\ a_1b_3 + a_3b_1 \\ -a_1b_2 - a_2b_1 \end{pmatrix}_{\mathbf{3}_2} . \end{aligned} \tag{675}$$

C Representations of A_4 in different basis

Here, we show another basis for representations of the A_4 group. First, we show the basis in section 4.1. All of the A_4 elements are written by products of the generators, s and t, which satisfy

$$s^2 = t^3 = (st)^3 = e \ . \tag{676}$$

On the representation **3**, these generators are represented as

$$s = a_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \quad t = b_1 = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}. \tag{677}$$

Next, we consider another basis, which is used, e.g. in Ref. [97]. In this basis, we denote the generators a and b, which correspond to s and t, respectively, and these generators are represented as

$$a = \frac{1}{3}\begin{pmatrix} -1 & 2 & 2 \\ 2 & -1 & 2 \\ 2 & 2 & -1 \end{pmatrix}, \quad b = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \omega^2 & 0 \\ 0 & 0 & \omega \end{pmatrix}, \tag{678}$$

on the representation **3**. These bases are transformed by the following unitary transformation matrix U_ω as

$$U_\omega = \frac{1}{\sqrt{3}}\begin{pmatrix} 1 & 1 & 1 \\ 1 & \omega & \omega^2 \\ 1 & \omega^2 & \omega \end{pmatrix}, \tag{679}$$

and the elements a and b are written as

$$a = U_\omega^\dagger s U_\omega = \frac{1}{3}\begin{pmatrix} -1 & 2 & 2 \\ 2 & -1 & 2 \\ 2 & 2 & -1 \end{pmatrix}, \quad b = U_\omega^\dagger t U_\omega = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \omega^2 & 0 \\ 0 & 0 & \omega \end{pmatrix}. \tag{680}$$

Therefore, the multiplication rule of the triplet is obtained as follows,

$$\begin{aligned} \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}_{\mathbf{3}} \otimes \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}_{\mathbf{3}} &= (a_1b_1 + a_2b_3 + a_3b_2)_{\mathbf{1}} \oplus (a_3b_3 + a_1b_2 + a_2b_1)_{\mathbf{1'}} \\ &\oplus (a_2b_2 + a_1b_3 + a_3b_1)_{\mathbf{1''}} \\ &\oplus \frac{1}{3}\begin{pmatrix} 2a_1b_1 - a_2b_3 - a_3b_2 \\ 2a_3b_3 - a_1b_2 - a_2b_1 \\ 2a_2b_2 - a_1b_3 - a_3b_1 \end{pmatrix}_{\mathbf{3}} \oplus \frac{1}{2}\begin{pmatrix} a_2b_3 - a_3b_2 \\ a_1b_2 - a_2b_1 \\ a_1b_3 - a_3b_1 \end{pmatrix}_{\mathbf{3}} . \end{aligned} \tag{681}$$